Whitney Multiapproximation

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Abstract

In this article we prove that Whitney theorem for the value of the best multiapproximation of a function $f \in L_p([a,b]^d)$, $0 < p < \infty$ by algebraic multipolynomial $p_{m-1}$ of degree $\leq m - 1$.

1. Introduction, definitions and main result

Whitney theorem has applications in many areas and has been further generalized to various classes of function and other approximating spaces.

Whitney theorem was proved by Burkill [1] when $(k = 2, p = \infty)$ and Storozhenko [2] when $(0 < p < 1)$.

In [3], [4] Whitney proved that if $f \in C([a,b])$ then $E_{k-1}(f)_{[a,b]} \leq W_k\omega_k\left(\frac{b-a}{k}, [a,b]\right)$ where $W_k = \text{const}$ depends only on $k$.

In 2003 E.S. Bhaya [5] proved the following theorem by using Whitney theorem of interpolatory type for $k$-monotone functions for K. A. Kopotun.

Theorem A: Let $m, k \in N, m < k$ and $f \in \Delta^k \cap W^m_p(I)$ . Then for any, $n \geq k - 1$, there exists an polynomial $p_n \in \Pi_n$ such that for any $p < 1$

$$\| f^{(j)} - p_n^{(j)} \|_p \leq c(p, k)\omega_{k-j}^p(f^{(j)}, n^{-1}, I)_p \quad \text{for} \ j = 1, \ldots, m.$$ 

In 2004 S.Dekel and D.Leviatan [6] proved the following Whitney estimate.
Theorem B: Given \( 0 < p \leq \infty, r \in \mathbb{N}, \) and \( d \geq 1, \) there exists a constant \( C(d, r, p), \) depending only on the three parameters, such that for every bounded convex domain \( \Omega \subset \mathbb{R}^d, \) and each function \( f \in L_p(\Omega), \)

\[
E_{r-1}(f, \Omega)_p \leq C(d, r, p)\omega_r(f, \text{diam}(\Omega), \Omega)_p,
\]

where \( E_{r-1}(f, \Omega)_p \) is the degree of approximation by polynomials of total degree \( r - 1, \) and \( \omega_r(f, \cdot)_p \) is the modulus of smoothness of order \( r. \)

In 2011 Dinh Dung and Tino Ullrich [7] proved the following Whitney type inequalities

Theorem C: Let \( 1 \leq p \leq \infty, r \in \mathbb{N}^d, \) then there is a constant \( C \) depending only on \( r, d \) such that for every \( f \in L_p(Q) \)

\[
\left( \sum_{e \in [d]} \prod_{i \in e} 2^{|e|} \right)^{-1} \Omega(f, \delta, Q)_{p, Q} \leq E_r(f)_{p, Q} \leq C\Omega(f, \delta, Q)_{p, Q},
\]

Where \( Q := [a_1, b_1] \times \ldots \times [a_d, b_d] \) and \( \delta = \delta(Q) := (b_1 - a_1, \ldots, b_d - a_d) \) is the size of \( Q. \)

For the proof our main result we need the following definitions:

Let us introduce a new version of Lagrange polynomial on \( \mathbb{R}^d, \) and call it a Lagrange multipolynomial.

**Definition 1.1.**

A Lagrange multipolynomial \( L(x, f) = L((x_1, x_2, \ldots, x_d); f) \)

\[
L((x_1, x_2, \ldots, x_d); f) = L((x_1, \ldots, x_d); f; (x_{01}, \ldots, x_{0d}), (x_{11}, \ldots, x_{1d}), \ldots, (x_{m1}, \ldots, x_{md}))
\]

(1)

that interpolates a function \( f \) at points \( x_0 = (x_{01}, \ldots, x_{0d}), x_1 = (x_{11}, \ldots, x_{1d}), \ldots, x_m = (x_{m1}, \ldots, x_{md}) \) (interpolation nodes) is defined as an algebraic multipolynomial of at most \( mth \) order that takes the same values at these points as the function \( f, \) that is

\[
L(x_i; f) = L((x_{i1}, \ldots, x_{id}); f) = f((x_{i1}, \ldots, x_{id})
\]

(2)

where \( i = 0, \ldots, m. \)

Example, for \( m = 1 \) we have
\[ L(x; f; x_0, x_1) = L((x_1, \ldots, x_d); f; (x_{01}, \ldots, x_{0d}), (x_{11}, \ldots, x_{1d})) \]

\[ = \frac{(x_1 - x_{11}) \ldots (x_d - x_{1d})}{(x_{01} - x_{11}) \ldots (x_{0d} - x_{1d})} f((x_{01}, \ldots, x_{0d})) + \frac{(x_1 - x_{01}) \ldots (x_d - x_{0d})}{(x_{11} - x_{01}) \ldots (x_{1d} - x_{0d})} f((x_{11}, \ldots, x_{1d})) \]

\[ = f(x_{01}, \ldots, x_{0d}) + \frac{f((x_{11}, \ldots, x_{1d})) - f((x_{01}, \ldots, x_{0d}))}{(x_{11} - x_{01}) \ldots (x_{1d} - x_{0d})} (x_1 - x_{01}) \ldots (x_d - x_{0d}) \] (3)

where \( x_{0j} \neq x_{1j} \), \( j = 1, \ldots, d \)

**Definition 1.2.**

Let \( l_k(x) = l_k((x_1, \ldots, x_d)) = l_k((x_1, \ldots, x_d); (x_{01}, \ldots, x_{0d}), \ldots, (x_{m1}, \ldots, x_{md})) \)

\[ = \prod_{i=0}^{m} \frac{(x_1 - x_{i1}) \ldots (x_d - x_{id})}{(x_{k1} - x_{i1}) \ldots (x_{kd} - x_{id})} , \quad k = 0, \ldots, m , \] (4)

a new version of fundamental Lagrange multipolynomials.

We set

\[ p(x) = p((x_1, \ldots, x_d)), x \in R^d \]

\[ = (x_1 - x_{01}) \ldots (x_d - x_{0d}) (x_1 - x_{11}) \ldots (x_d - x_{1d}) \ldots (x_1 - x_{m1}) \ldots (x_d - x_{md}) \].

And note that

\[ \dot{p}((x_{k1}, \ldots, x_{kd})) = \lim_{j=1,\ldots,d} \frac{p((x_1, \ldots, x_d))}{(x_1 - x_{k1}) \ldots (x_d - x_{kd})} \]

\[ = \lim_{j=1,\ldots,d} \prod_{i=0}^{m} ((x_1 - x_{i1}) \ldots (x_d - x_{id})) \]

\[ = \prod_{i=0}^{m} ((x_{k1} - x_{i1}) \ldots (x_{kd} - x_{id})) . \]

Therefore, for any \( k = 0, \ldots, m \), the new version of the fundamental Lagrange multipolynomials are represented in the form
\[ I_k((x_1, \ldots, x_d)) = I_k((x_1, \ldots, x_d); (x_{01}, \ldots, x_{0d}), (x_{11}, \ldots, x_{1d}), \ldots, (x_{m1}, \ldots, x_{md})) \]

\[ = \frac{p((x_1, \ldots, x_d))}{(x_1 - x_{k1}) \ldots (x_d - x_{kd})} \hat{p}((x_{k1}, \ldots, x_{kd})) , \]

where \( x_j \neq x_{kj} \), \( j = 1, \ldots, d \) , \( k = 0, \ldots, m \).

Let \( \delta_{i,k} \) denote the Kronecker symbol, which is equal to 1 for \( i = k \) and to 0 otherwise.

It follows from the obvious equality \( I_k(x_{i1}, \ldots, x_{id}) = \delta_{i,k} \), \( i, k = 0, \ldots, m \), that the Lagrange multipolynomial exists and is represented by the relation

\[ L((x_1, \ldots, x_d); f; (x_{01}, \ldots, x_{0d}), \ldots, (x_{m1}, \ldots, x_{md})) \]

\[ = \sum_{k=0}^{m} f((x_{k1}, \ldots, x_{kd})) I_k((x_1, \ldots, x_d); (x_{01}, \ldots, x_{0d}), \ldots, (x_{m1}, \ldots, x_{md})) \] (5)

**Definition 1.3.**

The expression \( [(x_{01}, \ldots, x_{0d}), (x_{11}, \ldots, x_{1d}), \ldots, (x_{m1}, \ldots, x_{md}); f] \) is called the divided difference of order \( m \) for the function \( f \) at the points \( x_0 = (x_{01}, \ldots, x_{0d}), x_1 = (x_{11}, \ldots, x_{1d}), \ldots, x_m = (x_{m1}, \ldots, x_{md}) \).

For example

\[ [x_0, x_1; f] = \frac{f((x_{01}, \ldots, x_{0d}))}{(x_{01} - x_{11}) \ldots (x_{0d} - x_{1d})} + \frac{f((x_{11}, \ldots, x_{1d}))}{(x_{11} - x_{01}) \ldots (x_{1d} - x_{0d})} \]

\[ = \frac{f((x_{01}, \ldots, x_{0d})) - f((x_{11}, \ldots, x_{1d}))}{(x_0 - x_{11}) \ldots (x_{0d} - x_{1d})} \] (6)

Let \( [x_0; f] = [(x_{01}, \ldots, x_{0d}); f] = f((x_{01}, \ldots, x_{0d})) \). (7)
Definition 1.4.

The expression

\[ \Delta_h^m (f; (x_{01}, \ldots, x_{0d})) := \sum_{k=0}^{m} \left( (-1)^{m-k} \binom{m}{k} \right)^d f((x_{01} + kh_1, \ldots, x_{0d} + kh_d)) \]  

(8)

where \( d \in \mathbb{N} \) chosen so that \( (-1)^{m-k} = (-1)^d \)

is called the multi \( m \)th difference of the function \( f \in L_p([a, b]^d), 0 < p < \infty \) at the point \( x_0 = (x_{01}, \ldots, x_{0d}) \) with step \( h = (h_1, \ldots, h_d) \).

Denote \( \Delta_h^0 (f; (x_{01}, \ldots, x_{0d})) = f((x_{01}, \ldots, x_{0d})) \) and \( \Delta_h^m (f; (x_{01}, \ldots, x_{0d})) = 0 \).

Our main result is:

Theorem 1.1.

If \( f \in L_p([a, b]^d), 0 < p < \infty \), then

\[ E_{m-1}(f)_{L_p([a, b]^d)} \leq C(p, m, d) \omega_m(f; [a, b]^d)_p \]

where \( h = (h_1, \ldots, h_d) \).

Now to prove our theorem we need the lemmas and theorems which will be stated and proved in the following sections:

2. Divided differences

Let us define the difference

\[ f((x_1, \ldots, x_d)) - L((x_1, \ldots, x_d); f; (x_{01}, \ldots, x_{0d}), \ldots, (x_{m1-1}, \ldots, x_{md-1})), \]

by the product \((x_1 - x_{01})(x_d - x_{0d}) \ldots (x_{1 - x_{m1-1}}) \ldots (x_{d - x_{md-1}})\)

Using (4) and (5), we represent the quotient at the points \( x_1 = x_{m1}, \ldots, x_d = x_{md} \) as follows:
Theorem 2.1. The Lagrange multipolynomial \( L(x; f; x_0, ..., x_m) \) is represented by the following Newton formula:

\[
L(x; f; x_0, ..., x_m) = L((x_1, ..., x_d); f; (x_{i_1}, ..., x_{i_d}), ..., (x_{m_1}, ..., x_{md}))
\]

\[
= [(x_{01}, ..., x_{0d}); f] + [(x_{01}, ..., x_{0d}), (x_{11}, ..., x_{1d}); f]((x_1 - x_{01}) ... (x_d - x_{0d})) + \cdots +
[(x_{01}, ..., x_{0d}), (x_{11}, ..., x_{1d}), ..., (x_{m_1}, ..., x_{md}); f]((x_1 - x_{01}) ... (x_d - x_{0d}))((x_1 - x_{11}) ... (x_d - x_{1d})) ... ((x_1 - x_{m-1}) ... (x_d - x_{md-1}))
\] (10)

Proof:

For \( m = 1 \), formula (10) follows from (3), (6) and (7).

Assume that (10) is true for a number \( m - 1 \).

By induction, let us prove that this formula is true for the number \( m \), that is

\[
L((x_1, ..., x_d); f; (x_{01}, ..., x_{0d}), ..., (x_{m1}, ..., x_{md}))
\]

\[
= L((x_1, ..., x_d); f; (x_{01}, ..., x_{0d}), ..., (x_{m1-1}, ..., x_{md-1})) + [(x_{01}, ..., x_{0d}), ..., (x_{m1}, ..., x_{md}); f]
((x_1 - x_{01}) ... (x_d - x_{0d})) ... ((x_1 - x_{m1-1}) ... (x_d - x_{md-1})).
\]

Since both parts of this equality are multipolynomials of degree \( \leq m \), it suffices to prove that this equality holds at all points \( x_i, i = 0, \ldots, m \).

By the definition of Lagrange multipolynomial (Definition 1.1), for all \( i = 0, \ldots, m - 1 \), we have

\[
L((x_{i1}, ..., x_{id}); f; (x_{01}, ..., x_{0d}), ..., (x_{m1-1}, ..., x_{md-1}))
\]
= L((x_{i1}, \ldots, x_{id}; f; (x_{01}, \ldots, x_{0d}), \ldots, (x_{m1}, \ldots, x_{md})) ,

for \( i = m \) according to (9) we obtain

\[
L((x_{m1}, \ldots, x_{md}); f; (x_{01}, \ldots, x_{0d}), \ldots, (x_{m1-1}, \ldots, x_{md-1})) + \]

\[
[(x_{01}, \ldots, x_{0d}), \ldots, (x_{m1}, \ldots, x_{md}); f] (x_{m1} - x_{01}) \ldots (x_{md} - x_{0d}) \ldots (x_{m1} - x_{m1-1}) \ldots (x_{md} - x_{md-1})
\]

\[
= f((x_{m1}, \ldots, x_{md})) = L((x_{m1}, \ldots, x_{md}); f; (x_{01}, \ldots, x_{0d}), \ldots, (x_{m1}, \ldots, x_{md})) \quad \square
\]

**Lemma 2.1.**

\[
L_{x_j}^m(f) = m! \psi \left[ (x_{01}, \ldots, x_{0d}), (x_{11}, \ldots, x_{1d}), \ldots, (x_{m1}, \ldots, x_{md}); f \right] \quad (11)
\]

where \( \psi \) is a constant and \( j = 1, \ldots, d \).

**Proof:**

We have

\[
((x_1 - x_{01})(x_1 - x_{11}) \ldots (x_1 - x_{m1-1}))(x_2 - x_{02})(x_2 - x_{12}) \ldots (x_2 - x_{m2-1}) \ldots (x_d - x_{0d})(x_d - x_{1d}) \ldots (x_d - x_{md-1})
\]

\[
= \left(x_j^m \prod_{i=0}^{m-1} \prod_{\ell \neq j} (x_\ell - x_{i\ell}) \right) + (c_1 x_j^{m-1} \prod_{i=0}^{m-1} \prod_{\ell \neq j} (x_\ell - x_{i\ell}) \right) + \cdots + (c_m \prod_{i=0}^{m-1} \prod_{\ell \neq j} (x_\ell - x_{i\ell}) \right),
\]
where \( c_1, c_2, \ldots, c_m \) are constant and \( j = 1, \ldots, d \)

\[
= m! \left( \prod_{i=0}^{m-1} \prod_{\ell \neq j}^d x_{i\ell} \right), \text{ so}
\]

\[
L_{x_j}^{(m)}(f) = m! \left( \prod_{i=0}^{m-1} \prod_{\ell \neq j}^d x_{i\ell} \right) \left[ (x_{01}, \ldots, x_{0d}, x_{11}, \ldots, x_{1d}, \ldots, (x_{m1}, \ldots, x_{md}); f) \right]
\]

\[
= m! \psi \left[ (x_{01}, \ldots, x_{0d}), (x_{11}, \ldots, x_{1d}), \ldots, (x_{m1}, \ldots, x_{md}); f \right],
\]

where \( \psi = \prod_{i=0}^{m-1} \prod_{\ell \neq j}^d (x_{i\ell} - x_{i\ell}) \) is a constant \( \square \)

**Lemma 2.2.**

The following identity is true

\[
(x_{0j} - x_{mj})[(x_{01}, \ldots, x_{0d}), \ldots, (x_{m1}, \ldots, x_{md}); f] = [(x_{0j}, \ldots, x_{0d}), \ldots, (x_{m1}, \ldots, x_{md}); f] - [(x_{1j}, \ldots, x_{1d}), \ldots, (x_{m1}, \ldots, x_{md}); f]
\]

(12)

where \( j = 1, \ldots, d \).

**Proof:**

Let \( L((x_1, \ldots, x_d)) = L((x_3, \ldots, x_d); f; (x_{01}, \ldots, x_{0d}), \ldots, (x_{m1}, \ldots, x_{md})) \).

It follows from (10) and (11) that

\[
L_{x_j}^{(m-1)}(f) = [(x_{0j}, \ldots, x_{0d}), \ldots, (x_{m1}, \ldots, x_{md}); f](m-1)! \psi +
\]

\[
[(x_{01}, \ldots, x_{0d}), \ldots, (x_{m1}, \ldots, x_{md}); f] \psi \left( m! \ x_j - (m - 1)! \ (x_{0j} + \ldots + x_{mj-1}) \right),
\]

Interchanging the points \( x_0 = (x_{01}, \ldots, x_{0d}) \) and \( x_m = (x_{m1}, \ldots, x_{md}) \) in (10) we get

\[
L_{x_j}^{(m-1)}(f) = [(x_{m1}, \ldots, x_{md}), (x_{11}, \ldots, x_{1d}), \ldots, (x_{m1}, \ldots, x_{md}); f](m-1)! \psi
\]

\[
+[(x_{m1}, \ldots, x_{md}), (x_{11}, \ldots, x_{1d}), \ldots, (x_{m1}, \ldots, x_{md-1}), (x_{01}, \ldots, x_{0d}); f] \]

\[
\psi \left( m! \ x_j - (m - 1)! \ (x_{mj} + x_{m1} + \ldots + x_{mj-1}) \right)
\]

\[
= [(x_{11}, \ldots x_{1d}), \ldots, (x_{m1}, \ldots, x_{md}); f](m-1)! \psi
\]

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Subtracting equalities

\[ [(x_{01}, ..., x_{0d}), ..., (x_{m1}, ..., x_{md}); f](m - 1)! \psi + [(x_{01}, ..., x_{0d}), ..., (x_{m1}, ..., x_{md}); f] \psi (m! x_j - (m - 1)! (x_{ij} + \cdots + x_{mj})) , \]

and

\[ [(x_{11}, ..., x_{1d}), ..., (x_{m1}, ..., x_{md}); f](m - 1)! \psi + [(x_{01}, ..., x_{0d}), ..., (x_{m1}, ..., x_{md}); f] \psi (m! x_j - (m - 1)! (x_{ij} + \cdots + x_{mj})) \],

we get

\[ ([(x_{01}, ..., x_{0d}), ..., (x_{m1-1}, ..., x_{md-1}); f] - [(x_{11}, ..., x_{1d}), ..., (x_{m1}, ..., x_{md}); f])(m - 1)! \psi - \left((m - 1)! \psi[[x_{01}, ..., x_{0d}), ..., (x_{m1}, ..., x_{md}); f](x_{0j} - x_{mj})] = 0. \]

By dividing on \((m - 1)! \psi\), we get

\[ (x_{0j} - x_{mj})[[x_{01}, ..., x_{0d}), ..., (x_{m1}, ..., x_{md}); f] \]

\[ = [(x_{01}, ..., x_{0d}), ..., (x_{m1-1}, ..., x_{md-1}); f] - [(x_{11}, ..., x_{1d}), ..., (x_{m1}, ..., x_{md}); f] \]

Now let \(x_0, x_1 \in [a, b]^d\) and let a function \(f\) be absolutely continuous on \([a, b]^d\). Then according to the Lebesgue theorem we have

\[ f((x_{11}, ..., x_{1d})) - f((x_{01}, ..., x_{0d})) = \int_{x_{0d}}^{x_{1d}} \cdots \int_{x_{01}}^{x_{11}} f_{t_1, t_2, ..., t_d} dt_1 ... dt_d . \]

Performing the change of variables \(t_1 = x_{01} + (x_{11} - x_{01}) \tilde{t}_1, \ldots, t_d = x_{0d} + (x_{1d} - x_{0d}) \tilde{t}_1\) we obtain

\[ [(x_{01}, ..., x_{0d}), (x_{11}, ..., x_{1d}); f] \]

\[ = \frac{1}{(x_{11} - x_{01}) \cdots (x_{1d} - x_{0d})} \int_{x_{01}}^{x_{11}} \cdots \int_{x_{0d}}^{x_{1d}} f_{t_1, t_2, ..., t_d} ((t_1, ..., t_d)) dt_1 ... dt_d \]

\[ = \int_0^1 \int_0^1 f'(((x_{01} + (x_{11} - x_{01}) \tilde{t}_1, ..., x_{0d} + (x_{1d} - x_{0d}) \tilde{t}_1)) d \tilde{t}_1^d , \]

where \(dt_1 = (x_{11} - x_{01}) d \tilde{t}_1 , \ldots, dt_d = (x_{1d} - x_{0d}) d \tilde{t}_1\), \(f' = f_{t_1, ..., t_d}\) and

\[ d \tilde{t}_1^d = d \tilde{t}_1 \cdot \tilde{t}_1 \cdots d \tilde{t}_1, \text{ d times}. \]
A similar representation is true for any \( m \) by virtue of the following theorem:

**Theorem 2.2.**

Let \( x_i \in [a, b]^d \) where \( x_i = (x_{i1}, \ldots, x_{id}) \) for \( i = 0, \ldots, m \).

If the function \( f \) has the absolute continuous \((m - 1)\)th derivative on \([a, b]^d\), then

\[
[(x_{01}, \ldots, x_{0d}), (x_{11}, \ldots, x_{1d}), \ldots, (x_{m1}, \ldots, x_{md}); f]
= \int_0^1 \ldots \int_0^1 \int_0^{\tilde{t}_1} \ldots \int_0^{\tilde{t}_{m-1}} \int_0^{\tilde{t}_{m-1}} f^{(m)}(x_{01}, \ldots, x_{0d})
+ ((x_{11} - x_{01})\tilde{t}_1, \ldots, (x_{1d} - x_{0d})\tilde{t}_1) + \ldots
+ ((x_{m1} - x_{m1-1})\tilde{t}_m, \ldots, (x_{md} - x_{md-1})\tilde{t}_m) \, d\tilde{t}_m^d \ldots d\tilde{t}_1^d
\tag{14}
\]

**Proof:**

Assume that representation (14) is true for a number \( m - 1 \). By induction, let us prove that (14) is also true for the number \( m \). Denote \( \tilde{t}_0 := 1 \). According to relation (12) and the induction hypothesis, we have

\[
(x_{mj} - x_{mj-1}) \left[(x_{01}, \ldots, x_{0d}), (x_{m1}, \ldots, x_{md}); f\right]
= [(x_{01}, \ldots, x_{0d}), \ldots, (x_{m1-2}, \ldots, x_{md-2}), (x_{m1}, \ldots, x_{md}); f]
- [(x_{01}, \ldots, x_{0d}), \ldots, (x_{m1-1}, \ldots, x_{md-1}); f]
= \int_0^{\tilde{t}_0} \ldots \int_0^{\tilde{t}_{m-2}} \int_0^{\tilde{t}_{m-2}} (\int_{u_1}^{v_1} \ldots \int_{u_d}^{v_d} f^{(m)}(t_1, \ldots, t_d) \, dt_1 \ldots dt_d) \, d\tilde{t}_m^d \ldots d\tilde{t}_1^d,
\]

where

\[
f^{(m)} = f_{t_1 \ldots t_1, t_2 \ldots t_2, \ldots, t_d \ldots t_d}^{m \text{ times}};
\]

\[
v_1 = x_{01} + \cdots + (x_{m1-2} - x_{m1-3})\tilde{t}_{m-2} + (x_{m1} - x_{m1-2})\tilde{t}_{m-1},
\]

\[
\vdots
\]

\[
v_d = x_{0d} + \cdots + (x_{md-2} - x_{md-3})\tilde{t}_{m-2} + (x_{md} - x_{md-2})\tilde{t}_{m-1},
\]

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It remains to introduce a new integration variable $\tilde{t}_m$ instead of $t = (t_1, \ldots, t_d)$ in the last integral by using the change of variables

\[ t_1 = x_{01} + (x_{11} - x_{01})\tilde{t}_1 + \cdots + (x_{m1} - x_{m1-1})\tilde{t}_{m-1} + (x_{m1} - x_{m1-1})\tilde{t}_m, \]

\[ t_d = x_{0d} + (x_{1d} - x_{0d})\tilde{t}_1 + \cdots + (x_{md} - x_{md-1})\tilde{t}_{m-1} + (x_{md} - x_{md-1})\tilde{t}_m. \]

And then note that this change of variables transforms the segment $[0, \tilde{t}_m]^d$ into the segment that connects the points $u = (u_1, \ldots, u_d)$ and $v = (v_1, \ldots, v_d)$.

**Lemma 2.3.**

Let $i \in N$, $i \leq m$ and let $x_i \in [a, b]^d$ then

\[ [(x_{01}, \ldots, x_{0d}), \ldots, (x_{m1}, \ldots, x_{md}); f] = [(x_{i1}, \ldots, x_{id}), \ldots, (x_{m1}, \ldots, x_{md}); f_i] \quad (15) \]

where $f_i((x_1, \ldots, x_d)) = [(x_{01}, \ldots, x_{0d}), \ldots, (x_{i1-1}, \ldots, x_{id-1}); (x_1, \ldots, x_d); f]$

**proof:**

Can easily be proved by induction with the use of (12)

**Lemma 2.4.**

Let $k \in N$, $k \leq m$, and let $x_i \in [a, b]^d$ for all $i = 0, \ldots, m$. If a function $f$ is $k$ times continuously differentiable on $[a, b]^d$ or $f$ has the $(k-1)\text{th}$ absolutely continuous derivative on $[a, b]^d$, then

\[ [(x_{01}, \ldots, x_{0d}), \ldots, (x_{m1}, \ldots, x_{md}); f] = \int_0^1 \cdots \int_0^1 [(x_{11}, \ldots, x_{1d}), \ldots, (x_{m1}, \ldots, x_{md}); f_{i1}^{d}] d\tilde{t}_1^d \quad (16) \]

**Proof:**

From (13), (14) and (15) we get
\[
\left[ (x_{01}, ..., x_{0d}), ..., (x_{m1}, ..., x_{md}); f \right]
\]
\[
= \int_0^1 ... \int_0^1 ... \int_0^{i_{k-1}} ... \int_0^{i_{k-1}} \left[ (x_{k1}, ..., x_{kd}), ..., (x_{m1}, ..., x_{md}); f_{i^d_1, ..., i^d_k} \right] \, d^d_i, \]

where

\[
f_{i^d_1, ..., i^d_k}(x_1, ..., x_d)
\]
\[
= f^{(k)} \left( (x_{01}, ..., x_{0d}) + (x_{11}, ..., x_{1d}) - (x_{01}, ..., x_{0d}) \right) i^d_1 + ... + \left( (x_{k1-1}, ..., x_{kd-1}) - (x_{k1-2}, ..., x_{kd-2}) \right) i^d_{k-1} + \left( (x_{k1}, ..., x_{kd}) - (x_{k1-1}, ..., x_{kd-1}) \right) i^d_k.
\]

In particular, if \( k = 1 \), then

\[
\left[ (x_{01}, ..., x_{0d}), ..., (x_{m1}, ..., x_{md}); f \right]
\]
\[
= \int_0^1 ... \int_0^1 \left[ (x_{11}, ..., x_{1d}), ..., (x_{m1}, ..., x_{md}); f_{i^d_1} \right] \, d^d_i. \quad \square
\]

3. Finite differences

In this section, we assume that the points \( x_i = (x_{i1}, ..., x_{id}) \) are equidistant, that is, for all \( i = 0, ..., m \) we have

\[
x_{i1} = x_{01} + ih_1, ..., x_{id} = x_{0d} + ih_d, \quad h \in \mathbb{R}^d, h_j \neq 0, j = 1, ..., d.
\]

For the Lagrange interpolation multipolynomial

\[
L((x_1, ..., x_d)) = L((x_1, ..., x_d); f; (x_{01}, ..., x_{0d}), ..., (x_{m1-1}, ..., x_{md-1}))
\]
\[
= \sum_{k=0}^{m-1} f((x_{k1}, ..., x_{kd})) L_k((x_1, ..., x_d), (x_{01}, ..., x_{0d}), ..., (x_{m1-1}, ..., x_{md-1})).
\]

We determine the values of the new version of the fundamental Lagrange multipolynomials \( L_k \) at the point \( x_1 = x_{m1}, ..., x_d = x_{md} \).

According to (4), we have
\[ I_k ((x_{m1}, ..., x_{md}), (x_{01}, ..., x_{0d}), ..., (x_{m1-1}, ..., x_{md-1})) \]
\[ = \prod_{i=0}^{m-1} \frac{(x_{m1} - x_{i1}) \cdots (x_{md} - x_{id})}{(x_{k1} - x_{i1}) \cdots (x_{kd} - x_{id})} \]
\[ = \prod_{i=0}^{m-1} \frac{(x_{01} + mh_1 - x_{01} - ih_1) \cdots (x_{0d} + mh_d - x_{0d} - ih_d)}{(x_{01} + kh_1 - x_{01} - ih_1) \cdots (x_{0d} + kh_d - x_{0d} - ih_d)} \]
\[ = \prod_{i=0}^{m-1} \frac{(m - i)h_1 \cdots (m - i)h_d}{(k - i)h_1 \cdots (k - i)h_d} \]
\[ = \prod_{i=0}^{m-1} \frac{(m - i) \cdots (m - i)}{(k - i) \cdots (k - i)} \]
\[ = \left( -(-1)^{m-k} \binom{m}{k} \right)^d \left( -(-1)^{m-k} \binom{m}{k} \right) \]

We represent the difference \( f((x_{m1}, ..., x_{md})) - L((x_{m1}, ..., x_{md})) \) in the form
\[ f((x_{m1}, ..., x_{md})) - L((x_{m1}, ..., x_{md})) = \sum_{k=0}^{m} \left( -(-1)^{m-k} \binom{m}{k} \right)^d f((x_{01} + kh_1, ..., x_{0d} + kh_d)). \] (17)

**Lemma 3.1.**

\[ \Delta_h^m (f; (x_{01}, ..., x_{0d})) \]
\[ = (m!)^d (h_1 \cdots h_d)^m [(x_{01}, ..., x_{0d}), (x_{01} + h_1, ..., x_{0d} + h_d), ..., (x_{01} + mh_1, ..., x_{0d} + mh_d); f], \] (18)

**Proof:**
Since $f((x_{m1}, \ldots, x_{md})) - L((x_{m1}, \ldots, x_{md}))$

$$= \sum_{k=0}^{m} \left( (-1)^{m-k} \binom{m}{k} \right)^d f((x_{01} + kh_1, \ldots, x_{0d} + kh_d))$$

and

$$f((x_{m1}, \ldots, x_{md})) - L((x_{m1}, \ldots, x_{md}); f; (x_{01}, \ldots, x_{0d}), \ldots, (x_{m1-1}, \ldots, x_{md-1}))$$

$$= \prod_{k=0}^{m-1} \left[ (x_{m1-1} - x_{k1}) \ldots (x_{md} - x_{kd}) \right] \left[ (x_{01}, \ldots, x_{0d}), (x_{11}, \ldots, x_{1d}), \ldots, (x_{m1}, \ldots, x_{md}); f \right].$$

then

$$\Delta_h^m (f; (x_{01}, \ldots, x_{0d}))$$

$$= \prod_{k=0}^{m-1} \left[ (x_{01} + mh_1 - x_{01} - kh_1) \ldots (x_{0d} + mh_d - x_{0d} - kh_d) \right] \left[ (x_{01}, \ldots, x_{0d}), (x_{01} + h_1, \ldots, x_{0d} + h_d), \ldots, (x_{01} + mh_1, \ldots, x_{0d} + mh_d); f \right]$$

$$= \prod_{k=0}^{m-1} \left[ ((m-k)h_1) \ldots ((m-k)h_d) \right] \left[ (x_{01}, \ldots, x_{0d}), (x_{01} + h_1, \ldots, x_{0d} + h_d), \ldots, (x_{01} + mh_1, \ldots, x_{0d} + mh_d); f \right]$$

$$= \left( mh_1 \right) \left( (m-1)h_1 \right) \left( (m-2)h_1 \right) \ldots \left( (m-m+1)h_1 \right) \ldots \left( mh_d \right) \left( (m-1)h_d \right) \left( (m-2)h_d \right) \ldots \left( (m-m+1)h_d \right) \left[ (x_{01}, \ldots, x_{0d}), (x_{01} + h_1, \ldots, x_{0d} + h_d), \ldots, (x_{01} + mh_1, \ldots, x_{0d} + mh_d); f \right]$$

$$= (m! h_1^m) \ldots (m! h_d^m) \left[ (x_{01}, \ldots, x_{0d}), (x_{01} + h_1, \ldots, x_{0d} + h_d), \ldots, (x_{01} + mh_1, \ldots, x_{0d} + mh_d); f \right]$$

$$= (m!)^d \left( h_1 \ldots h_d \right)^m \left[ (x_{01}, \ldots, x_{0d}), (x_{01} + h_1, \ldots, x_{0d} + h_d), \ldots, (x_{01} + mh_1, \ldots, x_{0d} + mh_d); f \right] \square$$

**Lemma 3.2.**

Let $x_0 \in [a, b]^d$, $h_j > 0$, $j = 1, \ldots, d$, $x_k = (x_{k1}, \ldots, x_{kd})$ such that

$x_{k1} = x_{01} + kh_1, \ldots, x_{kd} = x_{0d} + kh_d$ and $x_m \in [a, b]^d$, $x_m = (x_{m1}, \ldots, x_{md})$. 
If \( F \in L_p^1([a, b]^d) \), then for every \( x \in [a, b]^d \) the following inequality is true:

\[
\| F((x_1, \ldots, x_d)) - L((x_1, \ldots, x_d); F; (x_{01}, \ldots, x_{0d}), \ldots, (x_{m1}, \ldots, x_{md})) \|_{L_p([a, b]^d)} \leq C(p) \frac{1}{(m!)^d(h_1 \ldots h_d)^m} \|(x_1 - x_{01}) \ldots (x_d - x_{0d}) \ldots (x_1 - x_{m1}) \ldots (x_d - x_{md})\|_{L_p([a, b]^d)} \omega_m(F', h, [a, b]^d)_p
\]

(19)

**Proof:**

For every \( \tilde{t}_1 \in [0,1] \), we set

\[
F_{\tilde{t}_1}(u_1, \ldots, u_d) = F'(x_1, \ldots, x_d) + (u_1, \ldots, u_d) - (x_1, \ldots, x_d)) \tilde{t}_1
\]

\[
= F'(x_1 + (u_1 - x_1) \tilde{t}_1, \ldots, x_d + (u_d - x_d) \tilde{t}_1)
\]

where \( u \in [a, b]^d \).

Then relations (16) and (18) yield

\[
[(x_1, \ldots, x_d), (x_{01}, \ldots, x_{0d}), \ldots, (x_{m1}, \ldots, x_{md}); F]
\]

\[
= \int_0^1 \int_0^1 [(x_{01}, \ldots, x_{0d}), \ldots, (x_{m1}, \ldots, x_{md}); F_{\tilde{t}_1}] d \tilde{t}_1^d
\]

\[
= \frac{1}{(m!)^d(h_1 \ldots h_d)^m} \int_0^1 \int_0^1 \Delta^m_{\tilde{t}_1}(F_{\tilde{t}_1}; (x_{01}, \ldots, x_{0d})) d \tilde{t}_1^d
\]

Since

\[
\| \Delta^m_{\tilde{t}_1}(F_{\tilde{t}_1}; (u_1, \ldots, u_d)) \|_{L_p([a, b]^d)}
\]

\[
= \| \Delta^m_{\tilde{t}_1}(F'; (x_1 + (u_1 - x_1) \tilde{t}_1, \ldots, x_d + (u_d - x_d) \tilde{t}_1)) \|_{L_p([a, b]^d)}
\]

\[
\leq \omega_m(F', h \tilde{t}_1, [a, b]^d)_p \leq \omega_m(F', h, [a, b]^d)_p
\]

then

\[
\|[(x_1, \ldots, x_d), (x_{01}, \ldots, x_{0d}), \ldots, (x_{m1}, \ldots, x_{md}); F]\|_{L_p([a, b]^d)}
\]
= \left\| \frac{1}{(m!)^d (h_1 \ldots h_d)^m} \int_0^1 \ldots \int_0^1 \Delta_h^m \left( F_{t_1'; (x_0, \ldots, x_0d)} \right) \frac{dt_1}{L_p[a,b]} \right\| \\
\leq C(p) \frac{1}{(m!)^d (h_1 \ldots h_d)^m} \left\| \Delta_h^m \left( F_{t_1'; (x_0, \ldots, x_0d)} \right) \right\|_{L_p[a,b]} \\
= C(p) \frac{1}{(m!)^d (h_1 \ldots h_d)^m} \left\| \Delta_h^m \left( F'; (x_1 + (x_0 - x_1) \tilde{t}_1, \ldots, x_d + (x_0d - x_d) \tilde{t}_1) \right) \right\|_{L_p[a,b]} \\
\leq C(p) \frac{1}{(m!)^d (h_1 \ldots h_d)^m} \omega_m(F', h, [a,b]^d)_p,

relation (19) follows from (9) such that

\[ F((x_1, \ldots, x_d)) - L((x_1, \ldots, x_d); F; (x_0, \ldots, x_0d), \ldots, (x_{m1}, \ldots, x_{md})) = \left( (x_1 - x_01) \ldots (x_d - x_0d) \right) \ldots \left( (x_1 - x_{m1}) \ldots (x_d - x_{md}) \right) \left[ (x_1, \ldots, x_d), (x_01, \ldots, x_0d), \ldots, (x_{m1}, \ldots, x_{md}); F, \right] \]

since

\[ \left\| ((x_1, \ldots, x_d), (x_01, \ldots, x_0d), \ldots, (x_{m1}, \ldots, x_{md}); F) \right\|_{L_p[a,b]} \]

\[ \leq C(p) \frac{1}{(m!)^d (h_1 \ldots h_d)^m} \omega_m(F', h, [a,b]^d)_p, \]

then

\[ \left\| F((x_1, \ldots, x_d)) - L((x_1, \ldots, x_d); F; (x_0, \ldots, x_0d), \ldots, (x_{m1}, \ldots, x_{md})) \right\|_{L_p[a,b]} \]

\[ \leq C(p) \frac{1}{(m!)^d (h_1 \ldots h_d)^m} \left\| ((x_1 - x_01) \ldots (x_d - x_0d)) \ldots ((x_1 - x_{m1}) \ldots (x_d - x_{md})) \right\|_{L_p[a,b]} \omega_m(F', h, [a,b]^d)_p \]

\[ \square \]


Let \( x_k = (x_{k1}, \ldots, x_{kd}) \) and \( x_0 = (x_{01}, \ldots, x_{0d}) \) such that \( x_{0j} := a \),

\[ h_j := \frac{b - a}{m}, j = 1, \ldots, d \] and \( x_{k1} = x_{01} + kh_1, \ldots, x_{kd} = x_{0d} + kh_d \),
We fix \(x \in [a, b]^d\), choose \(\delta\) for which \((x + m\delta) \in [a, b]^d\) and let \(\delta' = (t_1\delta_1, \ldots, t_d\delta_d)\).

As a result, we get

\[
\int_0^1 \cdots \int_0^1 \Delta^m_{\delta'}(g; (x_1, \ldots, x_d)) \, dt_1 \ldots dt_d
\]

\[
= (-1)^{md} g((x_1, \ldots, x_d))
\]

\[
+ \sum_{k=1}^m (-1)^{m-k} \binom{m}{k}^d \int_0^1 \cdots \int_0^1 g((x_1 + kt_1\delta_1, \ldots, x_d + kt_d\delta_d)) \, dt_1 \ldots dt_d
\]

\[
= (-1)^{md} g((x_1, \ldots, x_d))
\]

\[
+ \sum_{k=1}^m (-1)^{m-k} \binom{m}{k}^d \int_0^1 \cdots \int_0^1 G'((x_1 + kt_1\delta_1, \ldots, x_d + kt_d\delta_d)) \, dt_1 \ldots dt_d
\]

\[
= (-1)^{md} g((x_1, \ldots, x_d))
\]

\[
+ \sum_{k=1}^m (-1)^{m-k} \binom{m}{k}^d \frac{1}{k^d(\delta_1 \ldots \delta_d)} \left( G((x_1 + k\delta_1, \ldots, x_d + k\delta_d)) - G((x_1, \ldots, x_d)) \right)
\]

whence

\[
|g((x_1, \ldots, x_d))| \\
\leq \int_0^1 \cdots \int_0^1 |\Delta^m_{\delta'}(g; (x_1, \ldots, x_d))| \, dt_1 \ldots dt_d
\]

\[
+ \sum_{k=1}^m (-1)^{m-k} \binom{m}{k}^d \frac{1}{k^d(\delta_1 \ldots \delta_d)} \left( G((x_1 + k\delta_1, \ldots, x_d + k\delta_d)) - G((x_1, \ldots, x_d)) \right)
\]
Then

\[
\|g\|_{L_p[a,b]^d} \leq \left\| \int_0^1 \ldots \int_0^1 |\Delta^m g| (g; (x_1, \ldots, x_d)) \right\| dt_1 \ldots dt_d \\
+ \sum_{k=1}^{m} \left( (-1)^{m-k} \binom{m}{k} \right)^{d} \frac{1}{k^d} G((x_1 + k \delta_1, \ldots, x_d + k \delta_d)) \\
- G((x_1, \ldots, x_d)) \right\|_{L_p[a,b]^d}
\]

\[
\leq C(p) \left\| \int_0^1 \ldots \int_0^1 |\Delta^m g| (g; (x_1, \ldots, x_d)) \right\| dt_1 \ldots dt_d \right\|_{L_p[a,b]^d} \\
+ C(p) \left\| \sum_{k=1}^{m} \left( (-1)^{m-k} \binom{m}{k} \right)^{d} \frac{1}{k^d} G((x_1 + k \delta_1, \ldots, x_d + k \delta_d)) \\
- G((x_1, \ldots, x_d)) \right\|_{L_p[a,b]^d}
\]

\[
\leq C(p) \omega_m (g, |\delta|, [a, b]^d)_p + C(p) \frac{2}{|\delta_1| \ldots |\delta_d|} \|G\|_{L_p[a,b]^d} \sum_{k=1}^{m} \left( \frac{m}{k} \frac{1}{k} \right)^{d} .
\]

By virtue of Lemma(3.2) we have \(\|G\|_{L_p[a,b]^d} \leq C(p) (h_1 \ldots h_d) \omega_m (F', h, [a, b]^d)_p\)

\[\]

\[= C(p) (h_1 \ldots h_d) \omega_m (f, h, [a, b]^d)_p.\]

Therefore

\[
E_{m-1}(f)_{L_p[a,b]^d} \leq \|g\|_{L_p[a,b]^d}
\]

\[
\leq C(p) \omega_m (g, |\delta|, [a, b]^d)_p + C(p) \frac{2}{|\delta_1| \ldots |\delta_d|} (h_1 \ldots h_d) \sum_{k=1}^{m} \left( \frac{m}{k} \frac{1}{k} \right)^{d} \omega_m (f, h, [a, b]^d)_p,
\]

note that \(\delta_j\) can always be chosen so that \(h_j \geq |\delta_j| \geq h_j / 2\), then \(E_{m-1}(f)_{L_p[a,b]^d} \leq C(p) \omega_m (f, h, [a, b]^d)_p\)

\[
+ C(p) \frac{2}{(h_1 \ldots h_d)} (h_1 \ldots h_d) \sum_{k=1}^{m} \left( \frac{m}{k} \frac{1}{k} \right)^{d} \omega_m (f, h, [a, b]^d)_p.
\]

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\[ C(p) + 2C(p) \sum_{k=1}^{m} \left( \frac{m}{k} \right)^d \omega_m(f, [a, b]^d) \]
\[ = C(p, m, d) \omega_m(f, [a, b]^d) , \]

where \( C(p, m, d) \) is Whitney constant

**References**


**الخلاصة**

برهنا في هذا البحث نظرية وتتي لأفضل تقريب متعدد للدالة \( f \) التي تنتمي إلى الفضاء \( L_p \) بواسطة متعددة الحدود الجبرية متعددة المتغيرات من الدرجة أقل أو تساوي 1 - \( m \).

الكلمات المفتاحية: نظرية وتتي، التقريب المتعدد، متعددة حدود لاكرانج.