

# Some Geometric Properties of Multivalent Convex Function for Operator on Hilbert Space

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## Abstract

By making use of the operator on Hilbert space, we introduce and study some properties of geometric of a subclass  $W\mathcal{K}_p(\alpha, \omega, \eta, \mathcal{T})$  of multivalent convex functions with negative coefficients.

Also we obtain some geometric properties, such as, coefficient inequality, growth and distortion theorem, extreme points, convex set, closure theorem, radius of close-to-convexity, weighted mean and inclusive properties.

## 1. Introduction

In this paper, the aim is to mention the basic facts about geometric function theory, which this is obtained from mixing of geometry and analysis. Its origin started from the 19th century, but it continued and continually applicable till now. Geometric Function Theory is an important branch of complex analysis; It deals with the geometric properties of the analytic functions. In particular, we will concentrate on the important ideas in this theory. The fundamentals of this theory are explained in most text books on this subject. Also, we review and consider the basic ideas, principles, definitions and the general principles of complex analysis, which underline the geometric function theory of a complex variable rather than the basic lemmas which are needed in the proofs of our results. A full discussion of these principles can be found in standard text books, [1], [2].

The study of multivalent functions is one of the main branches of geometric function theory and plays a central role in complex analysis.

Let  $W_p$  be the class of functions of the form:

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{n+p} z^{n+p} \quad (p \in \mathbb{N} = \{1, 2, \dots\}), \quad (1)$$

which are analytic and  $p$ -valent in the open unit disk  $\mathcal{U} = \{z \in \mathbb{C}: |z| < 1\}$ . Let  $\mathcal{K}_p$  denote the subclass of  $W_p$  consisting of functions of the form:

$$f(z) = z^p - \sum_{n=1}^{\infty} a_{n+p} z^{n+p} \quad (a_{n+p} \geq 0, p \in \mathbb{N} = \{1, 2, \dots\}), \quad (2)$$

**Definition (1.1):** A function  $f \in W_p$  is said to be in the class  $W\mathcal{K}_p(\alpha, \omega, \eta)$  if it satisfies

$$\left| \frac{zf''(z) + 2f'(z) - p(1+p)z^{p-1}}{\alpha(zf''(z) + 2f'(z) - p\omega) + p(p-\omega+1)} \right| < 1,$$

where  $0 \leq \dot{u} < 1, 0 \leq \dot{\omega} < P, 0 < \dot{\eta} \leq 1$  dan  $z \in \dot{U}$ .

Let  $H$  be a Hilbert space on the complex field. Let  $T$  be a linear operator on  $H$ . For a complex analytic function  $f$  on the unit disk  $\dot{U}$ , we denoted  $f(T)$ , the operator on  $H$  defined by the usual Riesz- Dunford integral [3]

$$f(T) = \frac{1}{2\pi i} \int_{\mathcal{C}} f(z)(zI - T)^{-1} dz,$$

where  $I$  is the identity operator on  $H$ ,  $\mathcal{C}$  is a positively oriented simple closed rectifiable contour lying in  $\dot{U}$  and containing the spectrum  $\kappa(T)$  of  $T$  in its interior domain [4]. Also  $f(T)$  can be defined by the series

$$f(T) = \sum_{\tilde{n}=0}^{\infty} \frac{f^{(\tilde{n})}(0)}{\tilde{n}!} T^{\tilde{n}},$$

which converges in the norm topology [5].

**Definition (1.2):** Let  $H$  be a Hilbert space and  $T$  be an operator on  $H$  such that  $T \neq \varphi$

and  $\|T\| < 1$ . Let  $\dot{u}, \dot{\omega}$  be real numbers such that  $0 \leq \dot{u} < 1, 0 \leq \dot{\omega} < P, 0 < \dot{\eta} \leq 1$ .

An analytic function  $f$  on the unit disk belong to the class  $W\mathcal{K}_P(\dot{u}, \dot{\omega}, \dot{\eta}, T)$  if it satisfy the inequality

$$\|Tf''(T) + 2f'(T) - P(1 + P)T^{P-1}\| < \dot{\eta} \|\dot{u}(Tf''(T) + 2f'(T) - P\dot{\omega}) + P(P - \dot{\omega} + 1)\|,$$

The operator on Hilbert space were consider recently by [6]-[12].

## 2. Main results:

The first theorem gives Coefficient inequality for a function  $f$  to be in the class  $W\mathcal{K}_P(\dot{u}, \dot{\omega}, \dot{\eta}, T)$ .

**Theorem (2.1):** Let  $f \in \mathcal{K}_P$  be defined by (2). Then  $f \in W\mathcal{K}_P(\dot{u}, \dot{\omega}, \dot{\eta}, T)$  for all  $T \neq \varphi$  if and only if

$$\sum_{\tilde{n}=1}^{\infty} (\tilde{n} + P)(\tilde{n} + P + 1)(1 + \dot{\eta}\dot{u})a_{\tilde{n}+P} \leq \dot{\eta}P(\dot{u} + 1)(P - \dot{\omega} + 1). \quad (3)$$

where  $0 \leq \dot{u} < 1, 0 \leq \dot{\omega} < P, 0 < \dot{\eta} \leq 1$ .

The result is sharp for the function  $f$  given by

$$f(z) = z^P - \frac{\dot{\eta}P(\dot{u}+1)(P-\dot{\omega}+1)}{(\tilde{n}+P)(\tilde{n}+P+1)(1+\dot{\eta}\dot{u})} z^{\tilde{n}+P}, \tilde{n} \geq 1. \quad (4)$$

**Proof:** Suppose that the inequality (3) holds. Then, we have

$$\begin{aligned} & \|Tf''(T) + 2f'(T) - P(1 + P)T^{P-1}\| - \dot{\eta} \|\dot{u}(Tf''(T) + 2f'(T) - P\dot{\omega}) + P(P - \dot{\omega} + 1)\| \\ & \left\| - \sum_{\tilde{n}=1}^{\infty} (\tilde{n} + P)(\tilde{n} + P + 1)a_{\tilde{n}+P} T^{\tilde{n}+P-1} \right\| \\ & - \dot{\eta} \left\| \dot{u}P(P + 1)T^{P-1} - \sum_{\tilde{n}=1}^{\infty} \dot{u}(\tilde{n} + P)(\tilde{n} + P + 1)a_{\tilde{n}+P} T^{\tilde{n}+P-1} - \dot{u}P\dot{\omega} + P(P - \dot{\omega} + 1) \right\| \\ & \leq \sum_{\tilde{n}=1}^{\infty} (\tilde{n} + P)(\tilde{n} + P + 1)(1 + \dot{\eta}\dot{u})a_{\tilde{n}+P} - \dot{\eta}P(\dot{u} + 1)(P - \dot{\omega} + 1) \leq 0. \end{aligned}$$

Hence,  $f \in W\mathcal{K}_P(\dot{u}, \dot{\omega}, \dot{\eta}, T)$ .

To show the converse, let  $f \in W\mathcal{K}_P(\dot{u}, \dot{\omega}, \dot{\eta}, T)$ . Then

$$\|Tf''(T) + 2f'(T) - P(1 + P)T^{P-1}\| < \dot{\eta} \|\dot{u}(Tf''(T) + 2f'(T) - P\dot{\omega}) + P(P - \dot{\omega} + 1)\|, \text{ gives}$$

$$\left\| - \sum_{\tilde{n}=1}^{\infty} (\tilde{n} + P)(\tilde{n} + P + 1)a_{\tilde{n}+P} T^{\tilde{n}+P-1} \right\|$$

$$< \eta \left\| \dot{u} P(P+1) \mathfrak{T}^{P-1} - \sum_{\tilde{n}=1}^{\infty} \dot{u}(\tilde{n}+P)(\tilde{n}+P+1) a_{\tilde{n}+P} \mathfrak{T}^{\tilde{n}+P-1} - \dot{u} P \dot{\omega} + P(P-\dot{\omega}+1) \right\|.$$

Setting  $\mathfrak{T} = \mathfrak{r}I$  ( $0 < \mathfrak{r} < 1$ ) in the above inequality, we get

$$\frac{\sum_{\tilde{n}=1}^{\infty} (\tilde{n}+P)(\tilde{n}+P+1) a_{\tilde{n}+P} \mathfrak{r}^{\tilde{n}+P-1}}{\dot{u} P(P+1) \mathfrak{r}^{P-1} - \sum_{\tilde{n}=1}^{\infty} \dot{u}(\tilde{n}+P)(\tilde{n}+P+1) a_{\tilde{n}+P} \mathfrak{r}^{\tilde{n}+P-1} - \dot{u} P \dot{\omega} + P(P-\dot{\omega}+1)} < \eta. \quad (5)$$

Upon clearing denominator in (5) and letting  $\mathfrak{r} \rightarrow 1$ , we obtain

$$\sum_{\tilde{n}=1}^{\infty} (\tilde{n}+P)(\tilde{n}+P+1) a_{\tilde{n}+P} < \eta P(\dot{u}+1)(P-\dot{\omega}+1) - \sum_{\tilde{n}=1}^{\infty} \eta \dot{u}(\tilde{n}+P)(\tilde{n}+P+1) a_{\tilde{n}+P}.$$

Thus

$$\sum_{\tilde{n}=1}^{\infty} (\tilde{n}+P)(\tilde{n}+P+1)(1+\eta \dot{u}) a_{\tilde{n}+P} \leq \eta P(\dot{u}+1)(P-\dot{\omega}+1),$$

which completes the proof.

**Corollary (2.1):** If  $f \in W\mathcal{K}_P(\dot{u}, \dot{\omega}, \eta, \mathfrak{T})$ , then

$$a_{\tilde{n}+P} \leq \frac{\eta P(\dot{u}+1)(P-\dot{\omega}+1)}{(\tilde{n}+P)(\tilde{n}+P+1)(1+\eta \dot{u})}, \tilde{n} \geq 1.$$

Next, we obtain the growth and distortion for a function  $f$  to be in the class  $W\mathcal{K}_P(\dot{u}, \dot{\omega}, \eta, \mathfrak{T})$ .

**Theorem (2.2):** If  $f \in W\mathcal{K}_P(\dot{u}, \dot{\omega}, \eta, \mathfrak{T})$ , and  $\|\mathfrak{T}\| < 1$ ,  $\mathfrak{T} \neq \emptyset$ , then

$$\|\mathfrak{T}\|^P - \frac{\eta P(\dot{u}+1)(P-\dot{\omega}+1)}{(P+1)(P+2)(1+\eta \dot{u})} \|\mathfrak{T}\|^{P+1} \leq \|f(\mathfrak{T})\| \leq \|\mathfrak{T}\|^P + \frac{\eta P(\dot{u}+1)(P-\dot{\omega}+1)}{(P+1)(P+2)(1+\eta \dot{u})} \|\mathfrak{T}\|^{P+1}$$

and

$$P \|\mathfrak{T}\|^{P-1} - \frac{\eta P(\dot{u}+1)(P-\dot{\omega}+1)}{(P+2)(1+\eta \dot{u})} \|\mathfrak{T}\|^P \leq \|f'(\mathfrak{T})\| \leq P \|\mathfrak{T}\|^{P-1} + \frac{\eta P(\dot{u}+1)(P-\dot{\omega}+1)}{(P+2)(1+\eta \dot{u})} \|\mathfrak{T}\|^P.$$

The result is sharp for the function  $f$  given by

$$f(z) = z^P - \frac{\eta P(\dot{u}+1)(P-\dot{\omega}+1)}{(P+1)(P+2)(1+\eta \dot{u})} z^{P+1}.$$

**Proof:** According to the Theorem (2.1), we get

$$\sum_{\tilde{n}=1}^{\infty} a_{\tilde{n}+P} \leq \frac{\eta P(\dot{u}+1)(P-\dot{\omega}+1)}{(P+1)(P+2)(1+\eta \dot{u})}.$$

Hence

$$\begin{aligned} \|f(\mathfrak{T})\| &\geq \|\mathfrak{T}\|^P - \sum_{\tilde{n}=1}^{\infty} a_{\tilde{n}+P} \|\mathfrak{T}\|^{\tilde{n}+P} \\ &\geq \|\mathfrak{T}\|^P - \|\mathfrak{T}\|^{P+1} \sum_{\tilde{n}=1}^{\infty} a_{\tilde{n}+P} \\ &\geq \|\mathfrak{T}\|^P - \frac{\eta P(\dot{u}+1)(P-\dot{\omega}+1)}{(P+1)(P+2)(1+\eta \dot{u})} \|\mathfrak{T}\|^{P+1}. \end{aligned}$$

Also,

$$\begin{aligned} \|f(\mathfrak{T})\| &\leq \|\mathfrak{T}\|^P + \sum_{\tilde{n}=1}^{\infty} a_{\tilde{n}+P} \|\mathfrak{T}\|^{\tilde{n}+P} \\ &\leq \|\mathfrak{T}\|^P + \frac{\eta P(\dot{u}+1)(P-\dot{\omega}+1)}{(P+1)(P+2)(1+\eta \dot{u})} \|\mathfrak{T}\|^{P+1}. \end{aligned}$$

In view of (Theorem 2.1), we have

$$\sum_{\tilde{n}=1}^{\infty} (\tilde{n} + P) a_{\tilde{n}+P} \leq \frac{\dot{\eta} P(\dot{\omega}+1)(P-\dot{\omega}+1)}{(P+2)(1+\dot{\eta}\dot{\omega})}.$$

Thus

$$\begin{aligned} \|f'(\mathfrak{U})\| &\geq P\|\mathfrak{U}\|^{P-1} - \sum_{\tilde{n}=1}^{\infty} (\tilde{n} + P) a_{\tilde{n}+P} \|\mathfrak{U}\|^{\tilde{n}+P-1} \\ &\geq P\|\mathfrak{U}\|^{P-1} - \|\mathfrak{U}\|^P \sum_{\tilde{n}=1}^{\infty} (\tilde{n} + P) a_{\tilde{n}+P} \\ &\geq P\|\mathfrak{U}\|^{P-1} - \frac{\dot{\eta} P(\dot{\omega}+1)(P-\dot{\omega}+1)}{(P+2)(1+\dot{\eta}\dot{\omega})} \|\mathfrak{U}\|^P \end{aligned}$$

and

$$\begin{aligned} \|f'(\mathfrak{U})\| &\leq P\|\mathfrak{U}\|^{P-1} - \|\mathfrak{U}\|^P \sum_{\tilde{n}=1}^{\infty} (\tilde{n} + P) a_{\tilde{n}+P} \\ &\leq P\|\mathfrak{U}\|^{P-1} - \frac{\dot{\eta} P(\dot{\omega}+1)(P-\dot{\omega}+1)}{(P+2)(1+\dot{\eta}\dot{\omega})} \|\mathfrak{U}\|^P. \end{aligned}$$

Therefore the proof is complete.

In the following, we prove extreme points of the class  $W\mathcal{K}_P(\dot{\omega}, \dot{\omega}, \dot{\eta}, \mathfrak{U})$ .

**Theorem (2.3):** Let  $f_0(z) = z^P$  and

$$f_{\tilde{n}}(z) = z^P - \frac{\dot{\eta} P(\dot{\omega}+1)(P-\dot{\omega}+1)}{(\tilde{n}+P)(\tilde{n}+P+1)(1+\dot{\eta}\dot{\omega})} z^{\tilde{n}+P}, \quad \tilde{n} \geq 1.$$

Then  $f \in W\mathcal{K}_P(\dot{\omega}, \dot{\omega}, \dot{\eta}, \mathfrak{U})$  if and only if it can be expressed in the form

$$f(z) = \sum_{\tilde{n}=0}^{\infty} B_{\tilde{n}} f_{\tilde{n}}(z), \quad (6)$$

where  $B_{\tilde{n}} \geq 0$  and  $\sum_{\tilde{n}=0}^{\infty} B_{\tilde{n}} = 1$ .

**Proof:** Assume that  $f$  can be expressed by (6). Then, we have

$$f(z) = \sum_{\tilde{n}=0}^{\infty} B_{\tilde{n}} f_{\tilde{n}}(z) = z^P - \sum_{\tilde{n}=0}^{\infty} \frac{\dot{\eta} P(\dot{\omega}+1)(P-\dot{\omega}+1)}{(\tilde{n}+P)(\tilde{n}+P+1)(1+\dot{\eta}\dot{\omega})} B_{\tilde{n}} z^{\tilde{n}+P}. \quad (7)$$

Thus

$$\sum_{\tilde{n}=0}^{\infty} \frac{(\tilde{n}+P)(\tilde{n}+P+1)(1+\dot{\eta}\dot{\omega})}{\dot{\eta} P(\dot{\omega}+1)(P-\dot{\omega}+1)} \frac{\dot{\eta} P(\dot{\omega}+1)(P-\dot{\omega}+1)}{(\tilde{n}+P)(\tilde{n}+P+1)(1+\dot{\eta}\dot{\omega})} B_{\tilde{n}} = \sum_{\tilde{n}=0}^{\infty} B_{\tilde{n}} = 1 - B_0 \leq 1, \quad (8)$$

and so  $f \in W\mathcal{K}_P(\dot{\omega}, \dot{\omega}, \dot{\eta}, \mathfrak{U})$ .

Conversely, Suppose that  $f$  given by (2) in the class  $W\mathcal{K}_P(\dot{\omega}, \dot{\omega}, \dot{\eta}, \mathfrak{U})$ . Then by

**Corollary (2.1)**, we have

$$a_{\tilde{n}+P} \leq \frac{\dot{\eta} P(\dot{\omega}+1)(P-\dot{\omega}+1)}{(\tilde{n}+P)(\tilde{n}+P+1)(1+\dot{\eta}\dot{\omega})}.$$

Setting

$$B_{\tilde{n}} = \frac{(\tilde{n}+P)(\tilde{n}+P+1)(1+\dot{\eta}\dot{\omega})}{\dot{\eta} P(\dot{\omega}+1)(P-\dot{\omega}+1)} a_{\tilde{n}+P}, \quad \tilde{n} \geq 1,$$

and  $B_{\tilde{n}} = 1 - \sum_{\tilde{n}=1}^{\infty} B_{\tilde{n}}$ . Then

$$f(z) = \sum_{\tilde{n}=0}^{\infty} B_{\tilde{n}} f_{\tilde{n}}(z),$$

This completes the proof of the theorem.

Next, we obtain the convex set of the class  $W\mathcal{K}_P(\acute{u}, \acute{\omega}, \acute{\eta}, \mathcal{T})$ .

**Theorem (2.4):** The class  $W\mathcal{K}_P(\acute{u}, \acute{\omega}, \acute{\eta}, \mathcal{T})$  is a convex set.

**Proof:** Let  $f_1$  and  $f_2$  be the arbitrary element of  $W\mathcal{K}_P(\acute{u}, \acute{\omega}, \acute{\eta}, \mathcal{T})$ . Then for every

$\forall (0 \leq \forall \leq 1)$ , we show that  $(1 - \forall) f_1 + \forall f_2 \in W\mathcal{K}_P(\acute{u}, \acute{\omega}, \acute{\eta}, \mathcal{T})$ . Thus, we have

$$(1 - \forall) f_1 + \forall f_2 = z^P - \sum_{\tilde{n}=1}^{\infty} ((1 - \forall) a_{\tilde{n}+P} + \forall b_{\tilde{n}+P}) z^{\tilde{n}+P}.$$

Hence,

$$\begin{aligned} & \sum_{\tilde{n}=1}^{\infty} (\tilde{n} + P)(\tilde{n} + P + 1)(1 + \acute{\eta}\acute{u})((1 - \forall) a_{\tilde{n}+P} + \forall b_{\tilde{n}+P}) \\ &= (1 - \forall) \sum_{\tilde{n}=1}^{\infty} (\tilde{n} + P)(\tilde{n} + P + 1)(1 + \acute{\eta}\acute{u}) a_{\tilde{n}+P} + \forall \sum_{\tilde{n}=1}^{\infty} (\tilde{n} + P)(\tilde{n} + P + 1)(1 + \acute{\eta}\acute{u}) b_{\tilde{n}+P} \\ &\leq (1 - \forall) \acute{\eta}P(\acute{u} + 1)(P - \acute{\omega} + 1) + \forall \acute{\eta}P(\acute{u} + 1)(P - \acute{\omega} + 1). \end{aligned}$$

This completes the proof.

We will consider the function  $f_{\hat{r}}(z)$  defined, for every  $\hat{r} = 1, 2, 3, \dots, \acute{\epsilon}$  by

$$f_{\hat{r}}(z) = z^P - \sum_{\tilde{n}=1}^{\infty} a_{\tilde{n}+P, \hat{r}} z^{\tilde{n}+P} \quad (a_{\tilde{n}+P, \hat{r}} \geq 0). \quad (9)$$

Next, we discuss the closure theorem.

**Theorem (2.5):** Let the function  $f_{\hat{r}}(z)$  defined by (9) be in the class  $W\mathcal{K}_P(\acute{u}, \acute{\omega}, \acute{\eta}, \mathcal{T})$ .

Then the function  $G(z)$  defined by

$$G(z) = \sum_{\hat{r}=1}^{\acute{\epsilon}} \hat{e}_{\hat{r}} f_{\hat{r}}(z) \text{ and } \sum_{\hat{r}=1}^{\acute{\epsilon}} \hat{e}_{\hat{r}} = 1, \hat{e}_{\hat{r}} \geq 0$$

is in the class  $W\mathcal{K}_P(\acute{u}, \acute{\omega}, \acute{\eta}, \mathcal{T})$ .

**Proof:** By using definition of  $G(z)$ , we get

$$G(z) = \left( \sum_{\hat{r}=1}^{\acute{\epsilon}} \hat{e}_{\hat{r}} \right) z^P - \sum_{\tilde{n}=1}^{\infty} \left( \sum_{\hat{r}=1}^{\acute{\epsilon}} \hat{e}_{\hat{r}} a_{\tilde{n}+P, \hat{r}} \right) z^{\tilde{n}+P},$$

further. Since  $f_{\hat{r}}(z)$  are in the class  $W\mathcal{K}_P(\acute{u}, \acute{\omega}, \acute{\eta}, \mathcal{T})$ , for every  $\hat{r} = 1, 2, 3, \dots, \acute{\epsilon}$ , we get

$$\sum_{\tilde{n}=1}^{\infty} (\tilde{n} + P)(\tilde{n} + P + 1)(1 + \acute{\eta}\acute{u}) a_{\tilde{n}+P, \hat{r}} \leq \acute{\eta}P(\acute{u} + 1)(P - \acute{\omega} + 1),$$

for every  $\hat{r} = 1, 2, 3, \dots, \acute{\epsilon}$ .

Hence, we can see that

$$\begin{aligned} & \sum_{\tilde{n}=1}^{\infty} (\tilde{n} + P)(\tilde{n} + P + 1)(1 + \acute{\eta}\acute{u}) \left( \sum_{\hat{r}=1}^{\acute{\epsilon}} \hat{e}_{\hat{r}} a_{\tilde{n}+P, \hat{r}} \right) \\ &= \sum_{\hat{r}=1}^{\acute{\epsilon}} \hat{e}_{\hat{r}} \left( \sum_{\tilde{n}=1}^{\infty} (\tilde{n} + P)(\tilde{n} + P + 1)(1 + \acute{\eta}\acute{u}) a_{\tilde{n}+P, \hat{r}} \right) \leq \sum_{\hat{r}=1}^{\acute{\epsilon}} \hat{e}_{\hat{r}} \acute{\eta}P(\acute{u} + 1)(P - \acute{\omega} + 1), \end{aligned}$$

which implies that  $G(z)$  is in the class  $W\mathcal{K}_P(\acute{u}, \acute{\omega}, \acute{\eta}, \mathcal{T})$ .

In the following, we prove the radius of close- to- convexity.

**Theorem (2.6):** Let the function  $f$  defined by (2) be in the class  $W\mathcal{K}_p(\acute{u}, \acute{o}, \acute{q}, \mathfrak{T})$ .

Then  $f$  is close-to-convex of order  $(0 \leq \check{\alpha} < 1)$  in  $|z| < \mathfrak{r}_1$ ,

where

$$\mathfrak{r}_1 = \inf_{\check{n} \geq 1} \left\{ \frac{(\check{n} + \mathfrak{p} + 1)(1 + \acute{q}\acute{u})}{\acute{q}(\acute{u} + 1)(\mathfrak{p} - \acute{o} + 1)} \right\}^{\frac{1}{\check{n}}} \quad (10)$$

The result is sharp, the external function given (4).

**Proof:** It is sufficient to show that

$$\left| \frac{f'(z)}{z^{\mathfrak{p}-1}} - \mathfrak{p} \right| \leq \mathfrak{p} \text{ for } |z| < \mathfrak{r}_1, \quad (11)$$

where  $\mathfrak{r}_1$  is given by (10). Indeed we find from (2) that

$$\begin{aligned} \left| \frac{f'(z)}{z^{\mathfrak{p}-1}} - \mathfrak{p} \right| &= \left| \frac{f'(z) - \mathfrak{p}z^{\mathfrak{p}-1}}{z^{\mathfrak{p}-1}} \right| = \left| \frac{\mathfrak{p}z^{\mathfrak{p}-1} - \sum_{\check{n}=1}^{\infty} (\check{n} + \mathfrak{p})a_{\check{n}+\mathfrak{p}}z^{\check{n}+\mathfrak{p}-1} - \mathfrak{p}z^{\mathfrak{p}-1}}{z^{\mathfrak{p}-1}} \right| \\ &= \left| \frac{-\sum_{\check{n}=1}^{\infty} (\check{n} + \mathfrak{p})a_{\check{n}+\mathfrak{p}}z^{\check{n}+\mathfrak{p}-1}}{z^{\mathfrak{p}-1}} \right| \leq \sum_{\check{n}=1}^{\infty} \frac{(\check{n} + \mathfrak{p})a_{\check{n}+\mathfrak{p}}|z|^{\check{n}}}{\mathfrak{p}} \leq 1, \end{aligned} \quad (12)$$

by using **Theorem (2.1)** and by (12) we get

$$\frac{(\check{n} + \mathfrak{p})}{\mathfrak{p}}|z|^{\check{n}} \leq \frac{(\check{n} + \mathfrak{p})(\check{n} + \mathfrak{p} + 1)(1 + \acute{q}\acute{u})}{\acute{q}\mathfrak{p}(\acute{u} + 1)(\mathfrak{p} - \acute{o} + 1)},$$

then

$$|z| \leq \left\{ \frac{(\check{n} + \mathfrak{p} + 1)(1 + \acute{q}\acute{u})}{\acute{q}(\acute{u} + 1)(\mathfrak{p} - \acute{o} + 1)} \right\}^{\frac{1}{\check{n}}}. \quad (13)$$

The result follows easily from (13).

**Definition (2.1):** Let  $f * \mathfrak{d}$  and  $\check{y} * \mathfrak{u}$  be in the class  $W\mathcal{K}_p(\acute{u}, \acute{o}, \acute{q}, \mathfrak{T})$ . Then, the

weighted mean  $Y_{\mathfrak{b}}$  of  $f * \mathfrak{d}$  and  $\check{y} * \mathfrak{u}$  given by

$$Y_{\mathfrak{b}}(z) = \frac{1}{2} \left( (1 - \mathfrak{b})(f * \mathfrak{d})(z) + (1 + \mathfrak{b})(\check{y} * \mathfrak{u})(z) \right), 0 < \mathfrak{b} < 1.$$

**Theorem (2.7):** Let  $f * \mathfrak{d}$  and  $\check{y} * \mathfrak{u}$  be in the class  $W\mathcal{K}_p(\acute{u}, \acute{o}, \acute{q}, \mathfrak{T})$ . Then, the

weighted mean of  $f * \mathfrak{d}$  and  $\check{y} * \mathfrak{u}$  is also in the class  $W\mathcal{K}_p(\acute{u}, \acute{o}, \acute{q}, \mathfrak{T})$ .

**Proof:** By using **Definition (2.1)**, we get

$$\begin{aligned} Y_{\mathfrak{b}}(z) &= \frac{1}{2} \left( (f * \mathfrak{d})(z) + (1 + \mathfrak{b})(\check{y} * \mathfrak{u})(z) \right) \\ &= \frac{1}{2} \left( (1 - \mathfrak{b}) \left[ z^{\mathfrak{p}} - \sum_{\check{n}=1}^{\infty} a_{\check{n}+\mathfrak{p}} s_{\check{n}+\mathfrak{p}} z^{\check{n}+\mathfrak{p}} \right] + (1 + \mathfrak{b}) \left[ z^{\mathfrak{p}} - \sum_{\check{n}=1}^{\infty} z_{\check{n}+\mathfrak{p}} \check{\delta}_{\check{n}+\mathfrak{p}} z^{\check{n}+\mathfrak{p}} \right] \right) \\ &= z^{\mathfrak{p}} - \sum_{\check{n}=1}^{\infty} \frac{1}{2} [(1 - \mathfrak{b})a_{\check{n}+\mathfrak{p}} s_{\check{n}+\mathfrak{p}} + (1 + \mathfrak{b})z_{\check{n}+\mathfrak{p}} \check{\delta}_{\check{n}+\mathfrak{p}}] z^{\check{n}+\mathfrak{p}}. \end{aligned}$$

Since  $f * \mathfrak{d}$  and  $\check{y} * \mathfrak{u}$  are in the class  $W\mathcal{K}_p(\acute{u}, \acute{o}, \acute{q}, \mathfrak{T})$  so by **Theorem (2.1)**, we have

$$\sum_{\tilde{n}=1}^{\infty} (\tilde{n} + \mathcal{P})(\tilde{n} + \mathcal{P} + 1)(1 + \eta\upsilon) a_{\tilde{n}+\mathcal{P}} \mathfrak{s}_{\tilde{n}+\mathcal{P}} \leq \eta\mathcal{P}(\upsilon + 1)(\mathcal{P} - \omega + 1)$$

and

$$\sum_{\tilde{n}=1}^{\infty} (\tilde{n} + \mathcal{P})(\tilde{n} + \mathcal{P} + 1)(1 + \eta\upsilon) z_{\tilde{n}+\mathcal{P}} \delta_{\tilde{n}+\mathcal{P}} \leq \eta\mathcal{P}(\upsilon + 1)(\mathcal{P} - \omega + 1).$$

Hence

$$\begin{aligned} & \sum_{\tilde{n}=1}^{\infty} (\tilde{n} + \mathcal{P})(\tilde{n} + \mathcal{P} + 1)(1 + \eta\upsilon) \left[ \frac{1}{2}(1 - \mathfrak{b}) a_{\tilde{n}+\mathcal{P}} \mathfrak{s}_{\tilde{n}+\mathcal{P}} + \frac{1}{2}(1 + \mathfrak{b}) z_{\tilde{n}+\mathcal{P}} \delta_{\tilde{n}+\mathcal{P}} \right] \\ &= \frac{1}{2}(1 - \mathfrak{b}) \sum_{\tilde{n}=1}^{\infty} (\tilde{n} + \mathcal{P})(\tilde{n} + \mathcal{P} + 1)(1 + \eta\upsilon) a_{\tilde{n}+\mathcal{P}} \mathfrak{s}_{\tilde{n}+\mathcal{P}} \\ &= \frac{1}{2}(1 + \mathfrak{b}) \sum_{\tilde{n}=1}^{\infty} (\tilde{n} + \mathcal{P})(\tilde{n} + \mathcal{P} + 1)(1 + \eta\upsilon) z_{\tilde{n}+\mathcal{P}} \delta_{\tilde{n}+\mathcal{P}} \\ &\leq \frac{1}{2}(1 - \mathfrak{b})\eta\mathcal{P}(\upsilon + 1)(\mathcal{P} - \omega + 1) + \frac{1}{2}(1 + \mathfrak{b})\eta\mathcal{P}(\upsilon + 1)(\mathcal{P} - \omega + 1). \end{aligned}$$

Which implies that  $\Upsilon_{\mathfrak{b}} \in W\mathcal{K}_{\mathcal{P}}(\upsilon, \omega, \eta, \mathfrak{T})$ .

Now, we obtain the inclusive properties of the class  $W\mathcal{K}_{\mathcal{P}}(\upsilon, \omega, \eta, \mathfrak{T})$ .

**Theorem (2.8):** Let  $0 \leq \upsilon < 1, 0 \leq \omega < \mathcal{P}, 0 < \eta < 1$ .

Then  $W\mathcal{K}_{\mathcal{P}}(\upsilon, \omega, \eta, \mathfrak{T}) \subset W\mathcal{K}_{\mathcal{P}}(\lambda, \omega, \frac{1}{3}, \mathfrak{T})$ , where

$$\lambda \leq \frac{1-2\eta\upsilon-3\eta}{\eta-1}.$$

**Proof:** Let the function  $f(\mathfrak{z})$  given by (2) belongs to the class  $W\mathcal{K}_{\mathcal{P}}(\upsilon, \omega, \eta, \mathfrak{T})$ .

Then in view of **Theorem (2.1)**, we get

$$\sum_{\tilde{n}=1}^{\infty} \frac{(\tilde{n}+\mathcal{P})(\tilde{n}+\mathcal{P}+1)(1+\eta\upsilon)}{\eta\mathcal{P}(\upsilon+1)(\mathcal{P}-\omega+1)} a_{\tilde{n}+\mathcal{P}} \leq 1. \quad (14)$$

We want to find the value  $\lambda$  such that

$$\sum_{\tilde{n}=1}^{\infty} \frac{(\tilde{n}+\mathcal{P})(\tilde{n}+\mathcal{P}+1)\left(1+\frac{1}{3}\lambda\right)}{\frac{1}{3}\mathcal{P}(\lambda+1)(\mathcal{P}-\omega+1)} a_{\tilde{n}+\mathcal{P}} \leq 1. \quad (15)$$

The inequality (14) would obviously imply (15) if

$$\frac{(\tilde{n}+\mathcal{P})(\tilde{n}+\mathcal{P}+1)\left(1+\frac{1}{3}\lambda\right)}{\frac{1}{3}\mathcal{P}(\lambda+1)(\mathcal{P}-\omega+1)} \leq \frac{(\tilde{n}+\mathcal{P})(\tilde{n}+\mathcal{P}+1)(1+\eta\upsilon)}{\eta\mathcal{P}(\upsilon+1)(\mathcal{P}-\omega+1)}.$$

Rewriting the inequality, we get

$$\lambda \leq \frac{1-2\eta\upsilon-3\eta}{\eta-1}.$$

This completes the proof.

## CONFLICT OF INTERESTS

There are no conflicts of interest.

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## بعض الخصائص الهندسية لدالة محدبة متعددة التكافؤ على فضاء هيلبرت

### الخلاصة

بالاستفادة من استعمال المؤثر على فضاء هيلبرت، نحنُ قدمنا ودرسنا بعض الخصائص الهندسية من الدوال التحليلية المتعددة التكافؤ المحدبة ذات معاملات سالبة.  $(\mathcal{H}_p(\alpha, \omega, \eta, \tau))$  للصنف الجزئي كذلك حصلنا على بعض الخصائص الهندسية، مثل، متراجحة المعاملات، مبرهنة النمو والتشوية، النقاط الحرجة، المجموعة المحدبة، نظرية الانغلاق، أنصاف الاقطار المغلقة، المتوسط الوزني وخواص الاحتواء لهذا الصنف.

**الكلمات المفتاحية:** الدوال المتعددة التكافؤ، فضاء هيلبرت، نظرية التشوية، انصاف اقطار المغلقة، النقاط الحرجة، نظرية الانغلاق والمتوسط الوزني.