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A Survey of Unmixed Bipartite Graphs and Very Well-Covered Graphs

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ABSTRACT

Many authors investigated unmixed and bipartite graphs. There are many results obtained on these two types of graphs and show that a bipartite graph G satisfies Serre's condition S_2, then G is Cohen-Macaulay. This paper is a survey with some new results about very well-covered graphs, an unmixed bipartite graph.

Keyword: Graphs, SNPs, Covered Graphs, unmixed, bipartite graphs.

INTRODUCTION

In this article, a simple graph without loops and multiple edges is called a graph. Let G be a graph with the vertex set V (G) = $\{x_1, ...x_n\}$and with the edge set E(G). The ideal I(G) and the quotient ring $k\{x_1, ...x_n\} \setminus I(G)$ are called the edge ideal of G and the edge ring of G, respectively. The simplicial complex of G is defined by Δ_G = A V A is an independent set in G}. A graph G is called unmixed if all the minimal vertex covers of G have the same number of elements and it is called well-covered [6] if all the maximal independent sets of G have the same number of elements. If G is a bipartite S_2 graph, then Gis Cohen-Macaulay. It is known that a group G in our class has a perfect matching [7].

We may assume that V (G) = X ∪ Y, X ∩ Y = φ ....(*). Where X = $\{x_1, ...x_n\}$ is a minimal Vertex cover of G and where Y = $\{y_1, ...y_n\} \subseteq E(G)$. Let G be a graph with 2_n vertices, which are not isolated, and with height I(G) = n. We assume condition (*). For any $i \in [n] := \{1, ..., n\}$, set $E_i := \{k \in [n] : x_k y_i \in E(G) \setminus \{i\}\}$, and define the graph $O_i(G)$ by

$O_i(G) := G - \{x_k y_i : k \in E_i\} + \{x_k x_i : k \in E_i\}$. Then, for every nonempty subset T := $\{i_1,\ldots\ldots, i_t\}$

of the set [n], we define $O_T(G) = O_{i1}O_{i2}\ldots\ldots O_{it}(G)$.

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1 RESULT'S CRITERION

1.1 unmixed and bipartite graphs

In this section, we survey unmixed graphs with bipartite and we present in this note the following combinatorial characterization of all the unmixed bipartite graphs and Cohen-Macaulay bipartite graphs.

Theorem 2.1. [1] Let G be a bipartite graph without isolated vertices. Then G is unmixed if and only if there is a bipartition $V_1 = \{x_1, ..., x_g\}$, $V_2 = \{y_1, ..., y_g\}$ of G such that: (a) $\{x_i, y_i\} \in E(G)$ for all i, and (b) if $\{x_i, y_i\}$ and $\{x_i, y_k\}$arein E(G) and i, j, k are distance, then $\{x_i, y_k\} \in E(G)$.

Proof. Since G is bipartite, there is a bipartition (V_1, V_2) of G, i.e., $V(G) = V_1 \cup V_2$, $V_1 \cup V2 = \varphi$, and every edge of G joins V_1 with V_2. Let g be the vertex covering the number of G, i.e., g is the number of elements in any minimal vertex cover of G. Notice that V_1 and V_2 are both minimal vertex covers of G,

hence $g = |V_1| = |V_2|$.

By K¨ onig theorem [3, Theorem 10.2, p. 96] g is the maximum number of independent edges of G. Therefore after permutation of the vertices we obtain that $V_1 = \{x_1, ..., x_g\}$, $V_2 = \{y_1, ..., y_{\}}$, and that $\{x_i, y_i\} \in E(G)$ for i = 1, ..., g. Thus we have proved that (a) holds.

To prove (b) take x_i, y_j and x_j, y_k in E(G) such that i, j, and k are distinct. Assume that x_i is not adjacent to y_k.

Then there is a maximal independent set of vertices A containing x_i and y_k. Notice that

A =g because G is unmixed. Hence

C = V (G) A is a minimal vertex cover of G with g vertices. Since x_i and y_k are not in C, we get that y_j and x_j are both in C. As C intersects$\{x_l, y_L\}$ in at least one vertex for

l = j, we obtain that | C |> g + 1, a contradiction.

Corollary 2.1. [1] Let G be a tree with at least three vertices. Then G is unmixed if and only if there is a bipartition $V_1 = \{x_1, ..., x_g\}$, $V_2 = \{y_1, ..., y_g\}$ of G such that: (a) $\{x_i, y_i\} \in E(G)$ for all i, and (b) for each i either $deg(x_i) = 1$ or $deg(y_i) = 1$.

Proposition 2.1. [2] Let G be an unmixed bipartite graph with bipartition $V = \{x_1, ..., x_n\}$

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And $W = \{y_1, ..., y_n\}$ of G such that:(a) x_i, y_i is an edge of G for all i =1,.......,n. Then V and W can be simultaneously relabeled such that the following statement are equivalent:

(a) There exist a linear order $V = F_0$, ...$F_n = W$ on some of the facets of Δ_G such that F_i and

$F_i + 1$ intersect in codimension one for i = 0, , n ̄1. (b)If $\{x_i, y_i\}$ is an edge, then $i \leq j$.

Proof. (a) → (b): We have $|F_1 \setminus F_0| = 1$, say $F_1 \setminus F_0 = \setminus\{y_1\}$. Then $F_1 = \{y_1, x_2, , x_n\}$ because$\{x_1, y_1\}$ is not a face of ΔG. Similarly,$|F_2 \setminus F - 1| = 1$, say $F_2 \setminus F_1 = \{y_2\}$. Thus $F_2 = \{y_1, y_2, x_3, , x_n\}$ because again $\{x_2, y_2\}$is not a face of ΔG. Hence by induction we may assume that $F_i = y_1, , y_i, x_i + 1, , x_n$ for i = 0, , n. In particular, if i > j, then x_i, y_j is a face of ΔG, and

hence it is not an edge of G.

(b) → (a): Set $F_i = \{y_1, , y_i, x_i+1, , x_n\}$. It is easy to see that for any i, F_i is a maximal independent set and hence a facet of ΔG. Moreover F_i and $F_i + 1$ intersect in codimension one.

Lemma 2.1. [3] Let G be an unmixed graph with non-isolated 2_n vertices and with height I(G) = n. Then G has a perfect matching.

Corollary 2.2. [3] Let G be an unmixed graph with 2_2 vertices, which are not isolated, and with height I(G) = n. We assume condition (*). Then,

(i) each minimal prime of I(G) is of the form (x_i, ...x_i k, y_ik+1, ..., y_in) where $\{x_i$, , in} =

{1,….., n};

(ii) $\{x_1 - y_1, ..., x_n - y_n\}$is a system of parameteres of S/I(G)

Proposition 2.2. [3]

Let G be a graph with 2_n vertices that are not isolated, and height I(G) = n. We assume condition (*). Then G is unmixed if and only if the following conditions hold.

(i) If $z_i x_i$, $y_j x_k \in E(G)$, then $z_i x_k \in E(G)$, for distinct i, i, k and for $z_i \in \{xi, y_i\}$

*(ii)*If $x_i y_i \in E(G)$, then $x_i x_j \in/ E(G)$.

Proposition 2.3. [2] Let G be an unmixed bipartite graph with bipartition $V = \{x_1$, ...$x_n\}$ and

$W = y\{_1, ...y_{n\}}$ such that x_i, y_i are edges for i = {1, ..., n}. Then V and W can be simultaneously relabeled such that the following statements are equivalent:

(a) There exists a linear order $V = F_0$, ...F_n = Won some of the facets of Δ(G)such that F_i and

$F_i + 1$ intersect in codimension one for I = 0, n–1.

(b) if $\{x_i, y_i\}$is an edge, then $i \leq j$.

Lemma 2.2. [2] Let G be an unmixed bipartite graph. Then G is a non-complete bipartite graph if and only if Δ(G) is connected.

Corollary 2.3. [5] Let G be a tree with at least four verities. Then the following are equivalent:

(a) G satisfies condition S_2.

(b) There is a bipartition $V = \{x_1, ..., x_n\}$, $W = \{y_1, ..., y_n\}$ of G such that

(i) $\{x_i, y_i\} \in E(G)$ for all i.

*(ii)*for each i either $deg(X_i = 1)$ or $deg(y_i) = 1$, exclusively.

(iii) V and W can be simultaneously relateted such that there exists an order

$V = F_0$, $F_n = w$

of the facets of Δ(G) where F_i and $F_i + n$ intersect in codimension one for i = 0, n–1.

Proposition 2.4. [2] let G be a bipartite S_2 graph. Let y be a vertices of degree one of G and x it's adjacent vertex. Then $G \setminus \{x, y\}$ is still an S_2 graph.

Corollary 2.4. [2] Let G be a tree with more than two vertices which is S_2. Let x be a degree one vertex of G and y its adjacent vertex. Then $G \setminus \{x, y\}$ is an S_2 graph.

Proposition 2.5. [3] Let G be an unmixed graph with 2n vertices, which are not isolated, and with height I(G) = n. We assume condition (*). Then the following conditions are equivalent.

(1) The subset $\{x_1y_1, x_2y_2, ..., x_ny_n\}$ of E(G) is unique perfect matching in G.

(2) The cycle C_{ij} is not a subgraph of G for any i < j.

(3) For any r > 2, the cycle $C_i1.......i_r$ is not a subgraph of G for any subset $\{i_1, i_2, ...i_r\} \subset [n]$

of cardinality r.

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1.2 Cohen-Macaulayness

In this section, we give combinatorial characterizations of Cohen-Macaulay graphs with unmixed.

Proposition 2.6. [3]

Let G be an unmixed graph with 2n vertices, which are not isolated, and with height I(G) = n. We assume condition (*). Then the following conditions are equivalent.

(1) G is Cohen-Macaulay.

(2) O_T (G) is Cohen-Macaulay for every subset T of [n].

(3) O_T (G) is unmixed for every subset T of [n].

Theorem 2.2. [2]

Let G be an unmixed bipartite graph with at least four vertices and with vertex partition V and W. Then the following are equivalent:

(a) G is unmixed and V and W can be labeled such that there exists an order V = F_0, F_n = W

of the facets of Δ(G) where F_1 and F_i + 1 intersects in codimension one for i = 0, n–1.

(b) G is the Cohen-Macaulay graph.

(c) G is a Buchsbaum non-complete bipartite graph.

(d) D is an S_2 graph.

Proof. We prove (a) → (b) → (c) → (d) → (a)

(a) → (b) Since G is unmixed, by K¨onig's Theorem there is a bipartition V = {x_1, , x_n} and W = {y_1, , y_n} such that x_i, y_i is an edge of G for all i. By Proposition 1.1, V and W can be relabeled such that x_i, y_i is an edge of G for all i and if x_i, y_j is an edge in G, then i j. We fix such labelling.

Let $\{x_i, y_j\}$ and $\{x_j, y_k\}$ be edges of G with $i < j < k$, and suppose that $\{x_i, y_k\}$ is not an edge of

G. Since $\{x_i, y_k\}$ is a face of Δ G and G is unmixed, ΔG is pure, hence there exists a facet F of ΔG with $|F| = n$ and $\{x_i, y_k\} \subset F$.

Since F is a facet of ΔG, any 2-element subset of F is a non-edge of G. We have $y_j \notin F$ since $\{x_i, y_j\}$ is an edge of G. Similarly $x_j \notin F$ since $\{x_j, y_k\}$ is an edge of G. On the other hand, since $\{x_t, y_t\}$ is an edge of G for all t, the facet F cannot contain both x_t and y_t. Hence F is of the form $F = \{z_1, , z_n\}$, where it $= x_t$ or y_t for $t = 1,\ldots, n$. Thus either y_j or x_j belongs to F, which is a contradiction. Consequently,

G is Cohen-Macaulay by the theorem of Herzog and Hibi.

(b) (c): Since every Cohen-Macaulay ring is a Buchsbaum ring, G is also Buchsbaum. By definition, the ideal of the simplicial complex ΔG is equal to the edge ideal of G. Hence ΔG is also Cohen-Macaulay and in particular, ΔG is connected. Therefore, by Lemma 1.2 G is non-complete.

(c) → (d): By [11, Corollary 2.7] the localization of every Buchsbaum ring at any of its prime ideals which is not equal to $(x_1,\ldots x_n, y_1,\ldots y_n)$, is Cohen-Macaulay. Therefore G satisfies the S_2 condition.

(d) → (a): Since ΔG satisfies the S_2 condition, by [4, Corollary 2.4] for any two facets F and H of

ΔG , there exist a positive integer m and a sequence $F = F_0, , F_m = H$ of facets of ΔG such that F_i

intersects $F_i + 1$ in codimension one for all $i = 0, , m_1$. Hence ΔG is strongly connected. In particular, since the partitions V and W of the vertices of G can be considered as two facets of ΔG and ΔG are strongly connected, the required sequence exists.

This implies that any two facets of ΔG have the same number of elements and hence G is unmixed.

Theorem 2.3. [3]

Let G be an unmixed graph with 2n vertices, which are not isolated, and with height I(G) = n. We assume condition (*). Then the following conditions are equivalent.

(1) G is Cohen-Macaulay.

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(2) $\Delta(G)$ is strongly connected

(3) The cycle C_ij is not a subgraph of G for any $i < j$.

Theorem 2.4. [3] Let G be an unmixed graph with 2n vertices, which are not isolated, and with height

I(G) = n. We assume conditions (*) and (**). Then the following conditions are equivalent.

(1) G is Cohen-Macaulay.

(2) G is unmixed.

(3) The following conditions hold:

(i) if $z_i x_j$, $y_j x_k \in E(G)$, then $z_i x_k \in E(G)$ for distinct i,j,k and for $z_i \in \{x_i, y_i\}$;

(ii) if $x_i y_i \in E(G)$, then $x_i x_j \in/ E(G)$.

Proof. (1) (2) This is well known.

$\rightarrow$ (2) $\overrightarrow{\ }$ (1)This follows from Theorem (2.3) since we assume condition (**).

$\rightarrow$ (2) (3)this follows from Proposition 2.2. we remark that the equivalence between (1) and (2) in Theorem (2.2) remark that the equivalence between (1) and (2) in Theorem (2.4) is a special case of [12].

Theorem 2.5. [4] Let G be a bipartite graph with vertex partition. Then the following condition is equivalent:

(a) G is a Cohen-Macaulay graph;

(b) | V |=| W | and the vertices $V = \{x_1, ...x_n\}$ and $W = \{y_1, ...y_n\}$can be labeled such that:

(i) $\{x_i, y_i\}$are edges for i = 1, , n;

(ii) if $\{x_i, y_i\}$is an edge, then $i \leq j$;

(iii) if $\{x_I, Y_j\}$ and $\{x_j, y_k\}$ are edges, then $\{x_i, y_k\}$ is also an edge.

Corollary 2.5. [2]

Every path of length more fantastic than four does not satisfy the condition S_2 and hence it is not Cohen- Macaulay.

Proof. By Corollary 2.3 every bipartiteS_2 graph has at least two vertices of degree one. From this fact and Theorem 1.3, we get the following result which is a special case of [[5], Proposition 6.2.1].

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1.3 very well covered graphs

Definition 2.1. [8]

Let G be a graph without isolated vertices and such that all maximal independent sets(equivalently, all minimal vertex covers) have the same cardinality. If furthermore, this cardinality is n, where 2_n is the number of vertices of G, then G is called very well covered.

Theorem 2.6. [8] Let G be a very well covered graph on 2_n vertices x_1, ...x_n, y_1, ..., y_n fulfilling conditions (*) and (**). Then ara I(G) = ht I(G) = n, i.e., I(G) is a set-theoretic complete intersection.

Corollary 2.6. [8] Let G be a very well covered graph. Then the following conditions are equivalent:

(1) R/I(G)is Cohen-Macaulay,

(2) I(G)is a set-theoretic complete intersection.

In particular, this equivalence is true if G is a bipartite graph.

Conflict of interests.

There are non-conflicts of interest.

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