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Almost Oscillatory Solutions of a Three-Dimensional Half-Linear Neutral differential System of the Second Order

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الحلول المتذبذبة على الاغلب لنظام تفاضلي محايد نصف خطي ثلاثي الأبعاد من الرتبة الثانية

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ABSTRACT

In this research, the oscillation and the asymptotic behavior of a half-linear three-dimensional neutral differential system of the second order have been studied, where all the non-oscillating solutions have been classified into 16 different classes, and then sufficient conditions were given to prove that most of these classes are inactive and non-occurring, that is empty, as for the rest classes, it has been proven that all its bounded solutions, either oscillating or non-oscillating, converge to zero when $t \to \infty$, and all unbounded solutions, are either oscillating or non-oscillating, goes to $\pm\infty$ as $t \to \infty$. Some examples are given to illustrate the obtained results.

Key words: Almost Oscillation, Neutral system, Three-dimensional, Second order.

الخلاصة

في هذا البحث تمت دراسة التذبذب والسلوك المقارب لنظام تفاضلي محايد ثلاثي الأبعاد نصف خطي من الدرجة الثانية ، حيث تم تصنيف جميع الحلول غير المتذبذبة إلى 16 فئة مختلفة ، ومن ثم تم إعطاء الشروط الكافية لإثبات ذلك. أن معظم هذه الفئات غير نشطة وغير متواجدة ، أي فارغة ، أما بالنسبة للفئات الباقية ، فقد ثبت أن جميع حلولها المحدودة ، سواء كانت متذبذبة أو غير متذبذبة ، تتقارب إلى الصفر عندما $t \to \infty$ ، وكلها الحلول الغير مقيدة ، اما ان تكون متذبذبة أو غير متذبذبة تذهب إلى $\pm\infty$ عندما $t \to \infty$. يتم إعطاء بعض الأمثلة لتوضيح النتائج التي تم الحصول عليها.

الكلمات المفتاحية: تذبذب تقريبا، نظام محايد ، ثلاثي الأبعاد ، ترتيب ثاني.

INTRODUCTION

Differential equations are one of the most important topics in mathematics due to their many applications [1-3]. NDEs represent one of these equations that have been of interest to many researchers, especially in the subject of oscillation and the behavior of the solution (see [4-8]). The oscillation properties of systems of differential equations, whether they are ordinary, delay, neutral, difference equations, or dynamic equations is the main concern of study (see [9-10]).
A few of these oscillation properties have been investigated in the system of NDEs, (see [11-13]).
Agarwal et al. [14], discussed the delay of nonlinear differential equation system as follows:

$$\begin{cases} \dot{x}(t) = a(t)f\big(y(t-\tau)\big) \\ \dot{y}(t) = -b(t)g\big(x(t-\tau)\big) \end{cases}$$

where they obtained sufficient conditions for the existence of nonoscillatory solution (NOS) for this system and established some sufficient conditions to insure the oscillations of the solutions for the system.
Mohamad and Abdulkareem,[15] discussed the almost oscillatory solutions of system of differential equations of the form:

$$\begin{cases} [r_1(t)([x(t)+p_1(t)x(\tau_1(t))]')^{\alpha}]' + q_1(t)f_1\big(y(\sigma_1(t))\big) = 0 \\ [r_2(t)([y(t)+p_2(t)y(\tau_2(t))]')^{\alpha}]' + q_2(t)f_2\Big(x\big(\sigma_2(t)\big)\Big) = 0 \end{cases}, \quad t \ge t_0.$$

In this work we consider the half-linear Neutral system of second order.

$$\begin{cases} \Big(\zeta_1(t)\big(\omega_1'(t)\big)^{\alpha_1}\Big)' = \lambda q_1(t) y_2^{\alpha_1}\big(\sigma_1(t)\big) \\ \Big(\zeta_2(t)\big(\omega_2'(t)\big)^{\alpha_2}\Big)' = \lambda q_2(t) y_3^{\alpha_2}\big(\sigma_2(t)\big), \qquad t \ge t_0 > 0 \\ \Big(\zeta_3(t)\big(\omega_3'(t)\big)^{\alpha_3}\Big)' = \lambda q_3(t) y_1^{\alpha_3}\big(\sigma_3(t)\big) \end{cases} \tag{1}$$

where,

$$\begin{cases} \omega_1(t) = y_1(t) + \mathcal{P}_1(t) y_1\big(\tau_1(t)\big), \\ \omega_2(t) = y_2(t) + \mathcal{P}_2(t) y_2\big(\tau_2(t)\big), \\ \omega_3(t) = y_3(t) + \mathcal{P}_3(t) y_3\big(\tau_3(t)\big). \end{cases} \tag{2}$$

for all $i = 1,2,3$, $\lambda \in \{1,-1\}$
The following hypotheses are assumed to be satisfied:
(H_1) $\alpha_i > 0$ is the ratio of two odd integers,
(H_2) $\zeta_i, q_i \in C([t_0,\infty),\mathbb{R}^+)$, $\mathcal{P}_i \in C([t_0,\infty),[0,1])$,
(H_3) $\tau_i \in C([t_0,\infty),\mathbb{R})$, $\tau_i(t) \le t$, and $\lim_{t\to\infty} \tau_i(t) = \infty$,
(H_4) $\sigma_i \in C([t_0,\infty),\mathbb{R})$, $\sigma_i(t) \le t$, and $\lim_{t\to\infty} \sigma_i(t) = \infty$.
By a solution to system (1), we mean functions $Y(t) = [y_1(t), y_2(t), y_3(t)]^T$ which have the properties $\zeta_i(t)\big(\omega_i'(t)\big)^{\alpha_i} \in C^1([t_0,\infty),\mathbb{R})$ and satisfy system (1).
Definition 1.1 A solution $Y(t) = [y_1(t), y_2(t), y_3(t)]^T$ of the system (1) is called proper if, $\sup\{|y_1(s)|, |y_2(s)|, |y_3(s)| : s \in [t,\infty)T\} > 0, t \ge t_0$.
Definition 1.2 A proper solution $Y(t)$ is said to oscillate if it is eventually trivial or if at least one component does not have an eventual constant sign. Otherwise, the solution is called nonoscillatory.
Definition 1.3 A solution $Y(t)$ of the system (1) is said to be almost oscillatory if some components of $Y(t)$ are oscillating and the others converge to zero.

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This paper consists of five sections: In the second and third sections, the asymptotic behavior of NOS with $\lambda = 1$ and $\lambda = -1$ respectively are studied. In the four sections, the main results for system (1) are mentioned, and finally, some examples to illustrate the main results are presented.

2. NOS of the System (1), Case $\lambda = 1$

In this section, the asymptotic behavior of NOS has studied with $\lambda = 1$ which will used in the following sections.

Remark 2.2. For simplicity, NOS is assumed here, if any exists, satisfy $y_1 y_2 y_3 > 0$.

Assume that K_j be the possible classes of the nature of NOS of (1), $j = 1,2, \ldots ,14$, where:

Table 1. The classes of all possible NOS of system (1), with $\lambda = 1$

Classes	**Sign of $\omega_i(t)$ and ω_i'**						**Sign of ω_i''**	**behavior as $t \to \infty$**
n	ω_1	ω_2	ω_3	ω_1'	ω_2'	ω_3'	ω_i''	ω_i
K_1	+	+	+	+	+	+	$\omega_1'' \geq 0$, $\omega_2'' \geq 0$, $\omega_3'' \geq 0$.	$\omega_i \to \infty$
K_2	+	+	+	−	−	−		$\zeta_i(\omega_i'(t))^{\alpha_i} \to 0$
K_3	+	+	+	−	−	+		$\omega_3 \to \infty$ $\zeta_1(\omega_1'(t))^{\alpha_1} \to 0$ $\zeta_2(\omega_2'(t))^{\alpha_2} \to 0$
K_4	+	+	+	−	+	−		$\omega_2 \to \infty$ $\zeta_1(\omega_1'(t))^{\alpha_1} \to 0$ $\zeta_3(\omega_3'(t))^{\alpha_3} \to 0$
K_5	+	+	+	+	−	−		$\omega_1 \to \infty$ $\zeta_2(\omega_2'(t))^{\alpha_2} \to 0$ $\zeta_3(\omega_3'(t))^{\alpha_3} \to 0$
K_6	+	+	+	+	+	−		$\omega_{1,2} \to \infty$ $\zeta_3(\omega_3'(t))^{\alpha_3} \to 0$
K_7	+	+	+	−	+	+		$\omega_{2,3} \to \infty$ $\zeta_1(\omega_1'(t))^{\alpha_1} \to 0$
K_8	+	+	+	+	−	+		$\omega_{1,3} \to \infty$ $\zeta_2(\omega_2'(t))^{\alpha_2} \to 0$
K_9	+	−	−	+	−	−	$\omega_1'' \leq 0$ $\omega_2'' \leq 0$	$\omega_2 \to -\infty$, $\omega_1 \to \infty$ or $\zeta_1(\omega_1'(t))^{\alpha_1} \to 0$, $\omega_3 \to -\infty$ or $\zeta_3(\omega_3'(t))^{\alpha_3} \to 0$.

							$\omega_3'' \geq 0$	
$\boldsymbol{K_{10}}$	$+$	$-$	$-$	$+$	$+$	$-$		$\omega_1 \to \infty$ or $\zeta_1(\omega_1'(t))^{\alpha_1} \to 0$, $\zeta_2(\omega_2'(t))^{\alpha_2} \to 0$ $\omega_3 \to -\infty$ or $\zeta_3(\omega_3'(t))^{\alpha_3} \to 0$.
$\boldsymbol{K_{11}}$	$-$	$-$	$+$	$-$	$-$	$+$	$\omega_1'' \leq 0$, $\omega_2'' \geq 0$, $\omega_3'' \leq 0$.	$\omega_1 \to -\infty$, $\omega_2 \to -\infty$, or $\zeta_2(\omega_2'(t))^{\alpha_2} \to 0$, $\omega_3 \to \infty$ or $\zeta_3(\omega_3'(t))^{\alpha_3} \to 0$.
$\boldsymbol{K_{12}}$	$-$	$-$	$+$	$+$	$-$	$+$		$\zeta_1(\omega_1'(t))^{\alpha_1} \to 0$, $\omega_2 \to -\infty$, or $\zeta_2(\omega_2'(t))^{\alpha_2} \to 0$, $\omega_3 \to \infty$ or $\zeta_3(\omega_3'(t))^{\alpha_3} \to 0$.
$\boldsymbol{K_{13}}$	$-$	$+$	$-$	$-$	$+$	$-$	$\omega_1'' \geq 0$, $\omega_2'' \leq 0$, $\omega_3'' \leq 0$.	$\omega_1 \to -\infty$ or $\zeta_1(\omega_1'(t))^{\alpha_1} \to 0$, $\omega_2 \to \infty$, or $\zeta_2(\omega_2'(t))^{\alpha_2} \to 0$, $\omega_3 \to -\infty$
$\boldsymbol{K_{14}}$	$-$	$+$	$-$	$-$	$+$	$+$		$\omega_1 \to -\infty$ or $\zeta_1(\omega_1'(t))^{\alpha_1} \to 0$, $\omega_2 \to \infty$, or $\zeta_2(\omega_2'(t))^{\alpha_2} \to 0$, $\zeta_3(\omega_3'(t))^{\alpha_3} \to 0$.

Lemma 2.2 Assume that $Y(t)$ is NOS of system (1) with $\lambda = 1$. If

$$\int_T^{\infty} \left(\frac{1}{\zeta_i(t)}\right)^{\frac{1}{\alpha_i}} dt = \infty\,, \quad T \geq t_0\,.\, i = 1,2,3 \tag{3}$$

Then $Y(t)$ belong to one of the classes $k_j, j = 1,2,\ldots,14$.

Proof: Suppose that $Y = (y_1(t), y_2(t), y_3(t))$ are NOS of (1) with $\lambda = 1$ then we have four cases to consider:

Case1. If y_1, y_2, y_3 are eventually positive solution of (1) then we get, for all $t \geq t_0$.

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$(\zeta_1(t)(\omega_1'(t))^{\alpha_1})' \geq 0$, $(\zeta_2(t)(\omega_2'(t))^{\alpha_2})' \geq 0$, $(\zeta_3(t)(\omega_3'(t))^{\alpha_3})' \geq 0$. That means $\zeta_1(t)(\omega_1'(t))^{\alpha_1}, \zeta_2(t)(\omega_2'(t))^{\alpha_2}, \zeta_3(t)(\omega_3'(t))^{\alpha_3}$ nondecreasing, hence there exists $t_1 \geq t_0$ such that $\zeta_1(t)(\omega_1'(t))^{\alpha_1}, \zeta_2(t)(\omega_2'(t))^{\alpha_2}$ and $\zeta_3(t)(\omega_3'(t))^{\alpha_3}$ are eventually positive or eventually negative. So there are eight sub-cases that can be discussed, which are:

Table 2. The eight possible cases that can occur in system (1),

i	**$\zeta_1(\omega_1'(t))^{\alpha_1} > 0$**	**$\zeta_2(\omega_2'(t))^{\alpha_2} > 0$**	**$\zeta_3(\omega_3'(t))^{\alpha_3} > 0$**	
ii.	$\zeta_1(\omega_1'(t))^{\alpha_1} < 0$	$\zeta_2(\omega_2'(t))^{\alpha_2} < 0$	$\zeta_3(\omega_3'(t))^{\alpha_3} < 0$	
iii.	$\zeta_1(\omega_1'(t))^{\alpha_1} < 0$	$\zeta_2(\omega_2'(t))^{\alpha_2} < 0$	$\zeta_3(\omega_3'(t))^{\alpha_3} > 0$	
iv.	$\zeta_1(\omega_1'(t))^{\alpha_1} < 0$	$\zeta_2(\omega_2'(t))^{\alpha_2} > 0$	$\zeta_3(\omega_3'(t))^{\alpha_3} < 0$	**$t \geq t_1$**
v.	$\zeta_1(\omega_1'(t))^{\alpha_1} > 0$	$\zeta_2(\omega_2'(t))^{\alpha_2} < 0$	$\zeta_3(\omega_3'(t))^{\alpha_3} < 0$	
vi.	$\zeta_1(\omega_1'(t))^{\alpha_1} > 0$	$\zeta_2(\omega_2'(t))^{\alpha_2} > 0$	$\zeta_3(\omega_3'(t))^{\alpha_3} < 0$	
vii.	$\zeta_1(\omega_1'(t))^{\alpha_1} < 0$	$\zeta_2(\omega_2'(t))^{\alpha_2} > 0$	$\zeta_3(\omega_3'(t))^{\alpha_3} > 0$	
viii.	$\zeta_1(\omega_1'(t))^{\alpha_1} > 0$	$\zeta_2(\omega_2'(t))^{\alpha_2} < 0$	$\zeta_3(\omega_3'(t))^{\alpha_3} > 0$	

Now, the sub-cases above will be discussed as follows:

i. $\zeta_1(t)(\omega_1'(t))^{\alpha_1} > 0, \zeta_2(t)(\omega_2'(t))^{\alpha_2} > 0, \ \zeta_3(t)(\omega_3'(t))^{\alpha_3} > 0$.

Since $\zeta_1(t)(\omega_1'(t))^{\alpha_1}$ is positive nondecreasing then there exists $t_2 \geq t_1$ and $c_1 > 0$ such that $\zeta_1(t)(\omega_1'(t))^{\alpha_1} \geq c_1, \ t \geq t_2$ therefore

$$\omega_1'(t) \geq \left(\frac{c_1}{\zeta_1(t)}\right)^{\frac{1}{\alpha_1}}, t \geq t_2. \tag{4}$$

Integrating (4) from t_2 to t we obtain,

$$\omega_1(t) - \omega_1(t_2) \geq c_1^{\frac{1}{\alpha}} \int_{t_2}^{t} \left(\frac{1}{\zeta_1(s)}\right)^{\frac{1}{\alpha_1}} ds$$

As $t \to \infty$, it follows that $\lim_{t\to\infty} \omega_1(t) = \infty$. Similarly we get $\lim_{t\to\infty} \omega_2(t) = \infty$ and $\lim_{t\to\infty} \omega_3(t) = \infty$ that is, $(\omega_1, \omega_2, \omega_3) \in K_1$

ii. $\zeta_1(t)(\omega_1'(t))^{\alpha_1} < 0, \ \zeta_2(t)(\omega_2'(t))^{\alpha_2} < 0, \ \zeta_3(t)(\omega_3'(t))^{\alpha_3} < 0, \ \ t \geq t_1$.

Suppose $\lim_{t\to\infty} \zeta_1(t)(\omega_1'(t))^{\alpha_1} = d_1 \leq 0$. We claim that $d_1 = 0$, otherwise $d_1 < 0$, then $\zeta_1(t)(\omega_1'(t))^{\alpha_1} \leq d_1 < 0$, that is

$$\omega_1'(t) \leq \left(\frac{d_1}{\zeta_1(t)}\right)^{\frac{1}{\alpha_1}}, \ \ t \geq t_2. \tag{5}$$

Integrating (5) from t_2 to t

$$\omega_1(t) - \omega_1(t_2) \leq d_1^{\frac{1}{\alpha_1}} \int_{t_2}^{t} \left(\frac{1}{\zeta_1(s)}\right)^{\frac{1}{\alpha_1}} ds.$$

as $t \to \infty$, $\lim_{t\to\infty} \omega_1(t) = -\infty$, a contradiction, Therefor $\lim_{t\to\infty} \zeta_1(t)(\omega_1'(t))^{\alpha_1} = 0$, similarly we can show that $\lim_{t\to\infty} \zeta_2(t)(\omega_2'(t))^{\alpha_2} = 0$ and $\lim_{t\to\infty} \zeta_3(t)(\omega_3'(t))^{\alpha_3} = 0$ hence $(\omega_1, \omega_2, \omega_3) \in K_2$.

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The proof of the **iii.-viii.** is similar and will be omitted. That leads to $(\omega_1,\omega_2,\omega_3)\in K_i, i=3,4,\ldots,8$, respectively.

Case 2. Let y_1 be an eventually positive solution and y_2, y_3 are eventually negative solution then from (1) we get $(\zeta_1(t)(\omega_1'(t))^{\alpha_1})'\le 0, (\zeta_2(t)(\omega_2'(t))^{\alpha_2})'\le 0, (\zeta_3(t)(\omega_3'(t))^{\alpha_3})'\ge 0$.

That means $\zeta_1(t)(\omega_1'(t))^{\alpha_1}$ and $\zeta_2(t)(\omega_2'(t))^{\alpha_2}$ are nonincreasing while $\zeta_3(t)(\omega_3'(t))^{\alpha_3}$ is nondecreasing. We have two subcases that can be discussed, which are:

v. $\boldsymbol{\zeta_1(t)(\omega_1'(t))^{\alpha_1}>0, \zeta_2(t)(\omega_2'(t))^{\alpha_2}<0, \zeta_3(t)(\omega_3'(t))^{\alpha_3}<0, t\ge t_1}$.

$\zeta_1(t)(\omega_1'(t))^{\alpha_1}>0$ and $(\zeta_1(t)(\omega_1'(t))^{\alpha_1})'\le 0$

suppose that,

$\lim_{t\to\infty}\zeta_1(t)(\omega_1'(t))^{\alpha_1}=d_1$, then $\zeta_1(t)(\omega_1'(t))^{\alpha_1}\ge d_1\ge 0$, that is

$$\omega_1{}'(t)\ge\left(\frac{d_1}{\zeta_1(t)}\right)^{\frac{1}{\alpha_1}}, t\ge t_2. \quad (6)$$

Integrating (6) from t_2 to t we get

$$\omega_1(t)-\omega_1(t_2)\ge d_1{}^{\frac{1}{\alpha_1}}\int_{t_2}^{t}\left(\frac{1}{\zeta_1(s)}\right)^{\frac{1}{\alpha_1}}ds$$

as $t\to\infty$, $\lim_{t\to\infty}\omega_1(t)=\infty$ or $\lim_{t\to\infty}\zeta_1(t)(\omega_1'(t))^{\alpha_1}=0$.

$\zeta_2(t)(\omega_2'(t))^{\alpha_2}<0$ and $(\zeta_2(t)(\omega_2'(t))^{\alpha_2})'\le 0$,

since $\zeta_2(t)(\omega_2'(t))^{\alpha_2}$ nonincreasing then there exists $t_2\ge t_1$ and $c_2<0$ such that

$\zeta_2(t)(\omega_2'(t))^{\alpha_2}\le c_2<0$,

$$\omega_2{}'(t)\le\left(\frac{c_2}{\zeta_2(t)}\right)^{\frac{1}{\alpha_2}}, \ t\ge t_2. \quad (7)$$

Integrating (7) from t_2 to t we get

$$\omega_2(t)-\omega_2(t_2)\le c_2{}^{\frac{1}{\alpha_2}}\int_{t_2}^{t}\left(\frac{1}{\zeta_2(s)}\right)^{\frac{1}{\alpha_2}}ds.$$

As $\to\infty$ $\lim_{t\to\infty}\omega_2(t)=-\infty$.

$\zeta_3(t)(\omega_3'(t))^{\alpha_3}<0$ and $(\zeta_3(t)(\omega_3'(t))^{\alpha_3})'\ge 0$

Suppose $\lim_{t\to\infty}\zeta_3(t)(\omega_3'(t))^{\alpha_3}=d_3\le 0$. We claim that $d_3=0$, otherwise $d_3<0$, then $\zeta_3(t)(\omega_3{}'(t))^{\alpha_3}\le d_3<0$, that is

$$\omega_3{}'(t)\le\left(\frac{d_3}{\zeta_3(t)}\right)^{\frac{1}{\alpha_3}}, t\ge t_2. \quad (8)$$

Integrating (8) from t_2 to t

$$\omega_3(t)-\omega_3(t_2)\le d_3{}^{\frac{1}{\alpha_3}}\int_{t_2}^{t}\left(\frac{1}{\zeta_3(s)}\right)^{\frac{1}{\alpha_3}}ds.$$

as $t\to\infty$, $\lim_{t\to\infty}\omega_3(t)=-\infty$, or $\lim_{t\to\infty}\zeta_3(t)(\omega_3'(t))^{\alpha_3}=0$ if $d_3=0$, that is $(\omega_1,\omega_2,\omega_3)\in K_9$

vi. $\boldsymbol{\zeta_1(t)(\omega_1'(t))^{\alpha_1}>0, \zeta_2(t)(\omega_2'(t))^{\alpha_2}>0, \zeta_3(t)(\omega_3'(t))^{\alpha_3}<0}$.

The proof this subcase leads to $(y_1,y_2,y_3)\in K_{10}$.

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Case 3. Let y_1 and y_2 are eventually negative solutions and y_3 be an eventually positive solution of system (1) with $\lambda = 1$. By applying the same proof style, followed in the above cases, taking into account the change that is obtained between variables and derivatives we get the following classes, $(\omega_1, \omega_2, \omega_3) \in K_{11}$ and $(\omega_1, \omega_2, \omega_3) \in K_{12}$.

***Case* 4**. Let y_1 and y_3 are eventually negative solutions and y_2 be an eventually positive solution of system (1) with $\lambda = 1$. By applying the same proof style, followed in the above cases, taking into account the change that obtains between variables and derivatives, the following classes are resulted in: $(\omega_1, \omega_2, \omega_3) \in K_{13}$ and $(\omega_1, \omega_2, \omega_3) \in K_{14}$.

3. NOS of System (1), Case $\lambda = -1$.

In this section, the asymptotic behavior of NOS is studied with $\boldsymbol{\lambda} = -1$ which will used in the following sections.

Lemma 3.1 Let $\lambda = -1$ and (3) hold. Assume that $Y(t) = \left(y_1(t), y_2(t), y_3(t)\right)$ are NOS of (1) Then $Y(t)$ belong to one of the following classes $\{L_i, i = 1,2,3, \ldots, 13\}$.

where:

Table 3. The classes of all possible NOS of system (1), with $\boldsymbol{\lambda} = \mathbf{-1}$.

Classes	**Sign of $\omega_i(t)$ and ω_i'**						**Sign of ω_i''**	**behavior as $t \to \infty$**
n	ω_1	ω_2	ω_3	ω_1'	ω_2'	ω_3'	ω_i''	ω_i
L_1	+	+	+	+	+	+	$\omega_i'' \le 0$	$\zeta_i(t)(\omega_i'(t))^{\alpha_i} \to$ 0or $\omega_i \to \infty$
L_2	+	−	−	−	−	−	$\omega_1'' \ge 0$, $\omega_2'' \ge 0$, $\omega_3'' \le 0$.	$\zeta_1(t)(\omega_1'(t))^{\alpha_1} \to 0$, $\omega_2 \to -\infty$ or $\zeta_2(t)(\omega_2'(t))^{\alpha_2} \to 0$ $\omega_3 \to -\infty$
L_3	+	−	−	−	−	+		$\zeta_1(t)(\omega_1'(t))^{\alpha_1} \to 0$, $\omega_2 \to -\infty$ or $\zeta_2(t)(\omega_2'(t))^{\alpha_2} \to 0$ $\zeta_3(t)(\omega_3'(t))^{\alpha_3} \to 0$
L_4	+	−	−	+	−	−		$\omega_1 \to \infty$ $\omega_2 \to -\infty$ or $\zeta_2(t)(\omega_2'(t))^{\alpha_2} \to 0$ $\omega_3 \to -\infty$
L_5	+	−	−	+	−	+		$\omega_1 \to \infty$ $\omega_2 \to -\infty$ or

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								$\zeta_2(t)(\omega_2'(t))^{\alpha_2} \to 0$ $\zeta_3(t)(\omega_3'(t))^{\alpha_3} \to 0$
$\boldsymbol{L_6}$	$-$	$-$	$+$	$-$	$-$	$-$	$\omega_1'' \geq 0,$ $\omega_2'' \leq 0,$ $\omega_3'' \geq 0.$	$\omega_1 \to -\infty$ or $\zeta_1(t)(\omega_1'(t))^{\alpha_1} \to 0,$ $\omega_2 \to -\infty,$ $\zeta_3(t)(\omega_3'(t))^{\alpha_3} \to 0.$
$\boldsymbol{L_7}$	$-$	$-$	$+$	$-$	$-$	$+$		$\omega_1 \to -\infty$ or $\zeta_1(t)(\omega_1'(t))^{\alpha_1} \to 0,$ $\omega_2 \to -\infty,$ $\omega_3 \to \infty$
$\boldsymbol{L_8}$	$-$	$-$	$+$	$-$	$+$	$-$		$\omega_1 \to -\infty$ or $\zeta_1(t)(\omega_1'(t))^{\alpha_1} \to 0,$ $\zeta_2(t)(\omega_2'(t))^{\alpha_2} \to 0$ $\zeta_3(t)(\omega_3'(t))^{\alpha_3} \to 0.$
$\boldsymbol{L_9}$	$-$	$-$	$+$	$-$	$+$	$+$		$\omega_1 \to -\infty$ or $\zeta_1(t)(\omega_1'(t))^{\alpha_1} \to 0,$ $\zeta_2(t)(\omega_2'(t))^{\alpha_2} \to 0$ $\omega_3 \to \infty.$
$\boldsymbol{L_{10}}$	$-$	$+$	$-$	$-$	$-$	$-$	$\omega_1'' \leq 0,$ $\omega_2'' \geq 0,$ $\omega_3'' \geq 0.$	$\omega_1 \to -\infty,$ $\zeta_2(t)(\omega_2'(t))^{\alpha_2} \to 0$ $\omega_3 \to -\infty$ or $\zeta_3(t)(\omega_3'(t))^{\alpha_3} \to 0.$
$\boldsymbol{L_{11}}$	$-$	$+$	$-$	$-$	$+$	$-$		$\omega_1 \to -\infty,$ $\omega_2 \to \infty,$ $\omega_3 \to -\infty$ or $\zeta_3(t)(\omega_3'(t))^{\alpha_3} \to 0.$
$\boldsymbol{L_{12}}$	$-$	$+$	$-$	$+$	$-$	$-$		$\zeta_1(t)(\omega_1'(t))^{\alpha_1} \to 0,$ $\zeta_2(t)(\omega_2'(t))^{\alpha_2} \to 0,$ $\omega_3 \to -\infty$ or $\zeta_3(t)(\omega_3'(t))^{\alpha_3} \to 0.$
$\boldsymbol{L_{13}}$	$-$	$+$	$-$	$+$	$+$	$-$		$\zeta_1(t)(\omega_1'(t))^{\alpha_1} \to 0,$ $\omega_2 \to \infty,$ $\omega_3 \to -\infty$ or $\zeta_3(t)(\omega_3'(t))^{\alpha_3} \to 0.$

Proof: Suppose that (1) with $\lambda = -1$ has NOS $(y_1(t), y_2(t), y_3(t))$, then we have four cases to consider:

***Case* 1.** Let y_1, y_2, y_3 are eventually positive solution of (1) then

$$(\zeta_i(t)(\omega_i'(t))^{\alpha_i})' \leq 0\,, i = 1,2,3\,, \qquad t \geq t_1 \geq t_0$$

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Which mean that $\zeta_1(t)(\omega_1'(t))^{\alpha_1}, \zeta_2(t)\left(\omega_2'(t)\right)^{\alpha_2}$ and $\zeta_3(t)\left(\omega_3'(t)\right)^{\alpha_3}$ are nonincreasing, so there is one sub-case that can be discussed, which are:

i. $\boldsymbol{\zeta_1(t)(\omega_1'(t))^{\alpha_1} > 0,\ \zeta_2(t)\left(\omega_2'(t)\right)^{\alpha_2} > 0,\ \zeta_3(t)\left(\omega_3'(t)\right)^{\alpha_3} > 0}$.

suppose $\lim_{t\to\infty} \zeta_1(t)(\omega_1'(t))^{\alpha_1} = d_1$, then $\zeta_1(t)(\omega_1'(t))^{\alpha_1} \geq d_1 \geq 0$, that is

$$\omega_1{}'(t) \geq \left(\frac{d_1}{\zeta_1(t)}\right)^{\frac{1}{\alpha_1}}, \quad t \geq t_2 \tag{9}$$

We claim that $d_1 = 0$, otherwise $d_1 > 0$.

Integrating (9) from t_2 to t we get,

$$\omega_1(t) - \omega_1(t_2) \geq d_1{}^{\frac{1}{\alpha_1}} \int_{t_2}^{t} \left(\frac{1}{\zeta_1(s)}\right)^{\frac{1}{\alpha_1}} ds$$

as $t \to \infty$ $\lim_{t\to\infty} \omega_1(t) = \infty$ or $\lim_{t\to\infty} \zeta_1(t)(\omega_1'(t))^{\alpha_1} = 0$ similarly we can get $\lim_{t\to\infty} \omega_2(t) = \infty$ or $\lim_{t\to\infty} \zeta_2(t)\left(\omega_2'(t)\right)^{\alpha_2} = 0$ and $\lim_{t\to\infty} \omega_3(t) = \infty$ or $\lim_{t\to\infty} \zeta_3(t)\left(\omega_3'(t)\right)^{\alpha_3} = 0$, that is $(y_1, y_2, y_3) \in L_1$.

Case 2. Let y_1 be an eventually positive solution and y_2, y_3 are eventually negative solution then from (1) we get,

$(\zeta_1(t)(\omega_1'(t))^{\alpha_1})' \geq 0\,, \left(\zeta_2(t)\left(\omega_2'(t)\right)^{\alpha_2}\right)' \geq 0, \left(\zeta_3(t)\left(\omega_3'(t)\right)^{\alpha_3}\right)' \leq 0\,.$

That means $\zeta_1(t)(\omega_1'(t))^{\alpha_1}$ and $\zeta_2(t)\left(\omega_2'(t)\right)^{\alpha_2}$ are nondecreasing while $\zeta_3(t)\left(\omega_3'(t)\right)^{\alpha_3}$ is nonincreasing. Hence, four subcases **(ii, iii, v, viii)** that can be discussed here, which are:

ii. $\boldsymbol{\zeta_1(t)(\omega_1'(t))^{\alpha_1} < 0,\ \zeta_2(t)\left(\omega_2'(t)\right)^{\alpha_2} < 0,\ \zeta_3(t)\left(\omega_3'(t)\right)^{\alpha_3} < 0}$.

$\zeta_1(t)(\omega_1'(t))^{\alpha_1} < 0$ and $(\zeta_1(t)(\omega_1'(t))^{\alpha_1})' \geq 0$

Suppose that $\lim_{t\to\infty} \zeta_1(t)(\omega_1'(t))^{\alpha_1} = d_1 \leq 0$. We claim that $d_1 = 0$, otherwise $d_1 < 0$, then $\zeta_1(t)(\omega_1'(t))^{\alpha_1} \leq d_1 < 0$, that is

$$\omega_1'(t) \leq \left(\frac{d_1}{\zeta_1(t)}\right)^{\frac{1}{\alpha_1}}, \ t \geq t_2. \tag{10}$$

Integrating (10) from t_2 to t

$$\omega_1(t) - \omega_1(t_2) \leq d_1{}^{\frac{1}{\alpha_1}} \int_{t_2}^{t} \left(\frac{1}{\zeta_1(s)}\right)^{\frac{1}{\alpha_1}} ds$$

$\lim_{t\to\infty} \omega_1(t) = -\infty$ a contradiction therefore $d_1 = 0$ and $\lim_{t\to\infty} \zeta_1(t)(\omega_1'(t))^{\alpha_1} = 0$.

$\zeta_2(t)\left(\omega_2'(t)\right)^{\alpha_2} < 0$ and $\left(\zeta_2(t)\left(\omega_2'(t)\right)^{\alpha_2}\right)' \geq 0$

Suppose that $\lim_{t\to\infty} \zeta_2(t)\left(\omega_2'(t)\right)^{\alpha_2} = d_2 \leq 0$. We claim that $d_2 = 0$, otherwise $d_2 < 0$, then $\zeta_2(t)\left(\omega_2'(t)\right)^{\alpha_2} \leq d_2 < 0$, that is

$$\omega_2'(t) \leq \left(\frac{d_2}{\zeta_2(t)}\right)^{\frac{1}{\alpha_2}}, \ t \geq t_2. \tag{11}$$

Integrating (11) from t_2 to t

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$$\omega_2(t)-\omega_2(t_2)\le d_2^{\frac{1}{\alpha_2}}\int_{t_2}^{t}\left(\frac{1}{\zeta_2(s)}\right)^{\frac{1}{\alpha_2}}ds,$$

as $t\to\infty$, $\lim_{t\to\infty}\omega_2(t)=-\infty$ or $\lim_{t\to\infty}\zeta_2(t)\left(\omega_2'(t)\right)^{\alpha_2}=0$ if $d_2=0$.

Since, $\zeta_3(t)\left(\omega_3'(t)\right)^{\alpha_3}<0$ and $\left(a_3(t)\left(\omega_3'(t)\right)^{\alpha_3}\right)'\le 0$.

That mean $\zeta_3(t)\left(\omega_3'(t)\right)^{\alpha_3}$ nonincreasing then there exists $t_2\ge t_1$ and $c_3<0$ such that $\zeta_3(t)\left(\omega_3'(t)\right)^{\alpha_3}\le c_3<0$

$$\omega_3'(t)\le\left(\frac{c_3}{\zeta_3(t)}\right)^{\frac{1}{\alpha_3}},\ t\ge t_2. \tag{12}$$

Integrating (12) from t_2 to t we get

$$\omega_3(t)-\omega_3(t_2)\le c_3^{\frac{1}{\alpha_3}}\int_{t_2}^{t}\left(\frac{1}{\zeta_3(s)}\right)^{\frac{1}{\alpha_3}}ds$$

As $\to\infty$ $\lim_{t\to\infty}\omega_3(t)=-\infty$ that is $(y_1,y_2,y_3)\in L_2$.

The proof of **iii, v,** and **viii.** are the same; hence, they are omitted. That lead to $(y_1,y_2,y_3)\in L_3$, $(y_1,y_2,y_3)\in L_4$ and $(y_1,y_2,y_3)\in L_5$ respectively.

***Case* 3**. Let y_1 and y_2 are eventually negative solution and y_3 be an eventually positive solution of system (1) with $\lambda=-1$. By applying the same proof style, followed in the above cases, taking into account the change that obtains the difference of variables and derivatives the following are obtained: $(y_1,y_2,y_3)\in L_6-L_9$.

***Case* 4**. If Let y_1 and y_3 are eventually negative solution y_2 is eventually positive solution of system (1) with $\lambda=-1$. By applying the same proof style, followed in the above cases which, taking into account the change that obtains the difference of variables and derivatives, the following, are achieved: $(y_1,y_2,y_3)\in L_{10}-L_{13}$.

4. Main Results for system (1)

In this section, some theorems and corollaries are presented to show that all bounded solutions oscillate or approach zero when time approaches infinity. In addition, deliver the behavior of some unbounded solutions under the same given conditions.

Theorem 4.1 Assume that $0<\mathcal{P}_j(t)<1, j=1,2,3, \lambda=-1,\ t\ge t_0$. In addition to the following conditions

$$\limsup_{t\to\infty}\int_T^t\left(\frac{1}{\zeta_1(s)}\int_s^{\delta(s)} q_1(s)\left(1-\mathcal{P}_2(\sigma_1(v))\right)^{\alpha_1}dv\right)^{\frac{1}{\alpha_1}}ds=\infty$$
$$\limsup_{t\to\infty}\int_{t_1}^t\left(\frac{1}{\zeta_2(s)}\int_s^{\delta(s)} q_2(s)\left(1-\mathcal{P}_3(\sigma_2(v))\right)^{\alpha_2}dv\right)^{\frac{1}{\alpha_2}}ds=\infty \quad (13)$$
$$\limsup_{t\to\infty}\int_{t_1}^t\left(\frac{1}{\zeta_3(s)}\int_s^{\delta(s)} q_3(s)\left(1-\mathcal{P}_1(\sigma_2(v))\right)^{\alpha_3}dv\right)^{\frac{1}{\alpha_2}}ds=\infty$$

Then every bounded solution of the system (1) oscillates.

Proof. Suppose that the system (1) has a NOS $Y(t)=(y_1(t),y_2(t),y_3(t))^{\mathrm{T}}$ so by Lemma 3.1, table 3, there is only the possible case $\boldsymbol{L}_1,\boldsymbol{L}_3,\boldsymbol{L}_8,\boldsymbol{L}_{12}$ to consider for $t\ge t_1\ge t_0$:

Case 1. Suppose that $Y(t)\in\boldsymbol{L}_1$. Then we have

$$\omega_i(t)=y_i(t)+\mathcal{P}_i(t)y_i(\tau_i(t))\Longrightarrow\omega_i(t)\ge y_i(t)$$

Then

$$y_i(t)\ge(1-\mathcal{P}_i(t))\omega_i(\tau_i(t)),$$

Or

$$y_i(\sigma_i(t))\ge\left(1-\mathcal{P}_i(\sigma_i(t))\right)\omega_i(\tau_i(\sigma_i(t)))$$

Integrating the first equation of system (1) from t to $\delta(t)$ to have

$$\zeta_1(\delta(t))(\omega_1'(\delta(t)))^{\alpha_1}-\zeta_1(t)(\omega_1'(t))^{\alpha_1}=-\int_t^{\delta(t)}q_1(s)y_2^{\alpha_1}(\sigma_2(s))\,ds$$
$$\zeta_1(t)(\omega_1'(t))^{\alpha_1}\ge\int_t^{\delta(t)}q_1(s)y_2^{\alpha_1}(\sigma_2(s))\,ds,$$
$$\zeta_1(t)(\omega_1'(t))^{\alpha_1}\ge\int_t^{\delta(t)}q_1(s)\left(1-\mathcal{P}_2(\sigma_2(s))\right)^{\alpha_1}\omega_2^{\alpha_1}(\tau_2(\sigma_2(s)))ds,$$
$$\omega_1'(t)\ge\omega_2(\tau_2(\sigma_2(t)))\left(\frac{1}{\zeta_1(t)}\int_t^{\delta(t)}q_1(s)\left(1-\mathcal{P}_2(\sigma_2(s))\right)^{\alpha_1}ds\right)^{\frac{1}{\alpha_1}}$$

Integrating the last inequality from t_1 to t to obtain

$$\omega_1(t)-\omega_1(t_1)\ge\int_{t_1}^t\omega_2(\tau_2(\sigma_2(s)))\left(\frac{1}{\zeta_1(s)}\int_s^{\delta(s)}q_1(s)\left(1-\mathcal{P}_2(\sigma_2(v))\right)^{\alpha_1}dv\right)^{\frac{1}{\alpha_1}}ds$$
$$\omega_1(t)-\omega_1(t_1)\ge\omega_2(\tau_2(\sigma_2(t_1)))\int_{t_1}^t\left(\frac{1}{\zeta_1(s)}\int_s^{\delta(s)}q_1(s)\left(1-\mathcal{P}_2(\sigma_1(v))\right)^{\alpha_1}dv\right)^{\frac{1}{\alpha_1}}ds$$

Letting $t\to\infty$ and by using the condition (13) leads to $\lim_{t\to\infty}\omega_1(t)=\infty$, implies that $\lim_{t\to\infty}y_1(t)=\infty$, a contradiction, similarly it can be shown that $\lim_{t\to\infty}y_2(t)=\infty$, $\lim_{t\to\infty}y_3(t)=\infty$, a contradiction this leads to the solutions $Y(t)=(y_1(t),y_2(t),y_3(t))^{\mathrm{T}}$ oscillates.

Case 2. Suppose that $Y(t)\in\boldsymbol{L}_3$. Then we have

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$$\omega_1(t) = y_1(t) + \mathcal{P}_1(t)y_1\big(\tau_1(t)\big) \Longrightarrow \omega_1(t) \ge y_1(t)$$
$$\omega_2(t) = y_2(t) + \mathcal{P}_2(t)y_2\big(\tau_2(t)\big) \Longrightarrow \omega_2(t) \le y_2(t)$$
$$\omega_3(t) = y_3(t) + \mathcal{P}_3(t)y_3\big(\tau_3(t)\big) \Longrightarrow \omega_3(t) \le y_3(t)$$

Then

$$\begin{aligned} &y_1(t) \ge \omega_1(t) - \mathcal{P}_1(t)\omega_1\big(\tau_1(t)\big) && (a)\\ &y_2\big(\sigma_2(t)\big) \le \Big(1-\mathcal{P}_2\big(\sigma_2(t)\big)\Big)\,\omega_2\big(\tau_2\big(\sigma_2(t)\big)\big) && (b)\\ &y_3\big(\sigma_2(t)\big) \le \Big(1-\mathcal{P}_3\big(\sigma_3(t)\big)\Big)\,\omega_3(\tau_3(\sigma_3(t))) && (c)\end{aligned} \qquad (14)$$

Integrating the first equation of system (1) from t to $\delta(t)$ and using (14b) to get

$$\zeta_1\big(\delta(t)\big)(\omega_1'(\delta(t)))^{\alpha_1} - \zeta_1(t)(\omega_1'(t))^{\alpha_1} = -\int_t^{\delta(t)} q_1(s)y_2^{\alpha_1}\big(\sigma_2(s)\big)\,ds$$

$$\omega_1'(t) \le \omega_2\big(\tau_2(\sigma_2(t)\big)\left(\frac{1}{\zeta_1(t)}\int_t^{\delta(t)} q_1(s)\,\Big(1-\mathcal{P}_2\big(\sigma_2(s)\big)\Big)^{\alpha_1} ds\right)^{\frac{1}{\alpha_1}}$$

Integrating the last inequality from t_1 to t to get

$$\omega_1(t) - \omega_1(t_1) \le \omega_2\big(\tau_1(\sigma_2(t_1)\big)\int_{t_1}^{t}\left(\frac{1}{\zeta_1(s)}\int_s^{\delta(s)} q_1(s)\,\Big(1-\mathcal{P}_2\big(\sigma_2(v)\big)\Big)^{\alpha_1} dv\right)^{\frac{1}{\alpha_1}} ds \qquad (15)$$

Letting $t \to \infty$ and in view of condition (13), the last inequality (15) leads to $\lim_{t\to\infty} \omega_1(t) = -\infty$, which is a contradiction since $\omega_1(t) > 0$ and bounded.

Integrating the second equation of system (1) from t to $\delta(t)$ and using (14c) to get

$$\zeta_2\big(\delta(t)\big)(\omega_2'(\delta(t)))^{\alpha_2} - \zeta_2(t)(\omega_2'(t))^{\alpha_2} = -\int_t^{\delta(t)} q_2(s)y_3^{\alpha_2}\big(\sigma_2(s)\big)\,ds$$

$$\omega_2'(t) \le \omega_3(\tau_3(\sigma_3(t)))\left(\frac{1}{\zeta_2(t)}\int_t^{\delta(t)} q_2(s)\,\Big(1-\mathcal{P}_3\big(\sigma_2(s)\big)\Big)^{\alpha_2} ds\right)^{\frac{1}{\alpha_2}}$$

Integrating the last inequality from t_1 to t to get

$$\omega_2(t) - \omega_2(t_1) \le \omega_3\Big(\tau_3\big(\sigma_3(t_1)\big)\Big)\int_{t_1}^{t}\left(\frac{1}{\zeta_2(s)}\int_s^{\delta(s)} q_2(s)\,\Big(1-\mathcal{P}_3\big(\sigma_2(v)\big)\Big)^{\alpha_2} dv\right)^{\frac{1}{\alpha_2}} ds \qquad (16)$$

Letting $t \to \infty$ and by using the condition (13) leads to $\lim_{t\to\infty} \omega_2(t) = -\infty$, implies that $\lim_{t\to\infty} y_2(t) = -\infty$. a contradiction since $y_2(t)$ is bounded.

Integrating the third equation of system (1) from t to $\delta(t)$ to get

$$\zeta_3\big(\delta(t)\big)(\omega_3'(\delta(t)))^{\alpha_3} - \zeta_3(t)(\omega_3'(t))^{\alpha_3} = -\int_t^{\delta(t)} q_3(s)y_1^{\alpha_3}\big(\sigma_1(s)\big)\,ds$$

$$\omega_3'(t) \ge \left(\frac{1}{\zeta_3(t)}\int_t^{\delta(t)} q_3(s)\,y_1^{\alpha_3}(\sigma_3(s))ds\right)^{\frac{1}{\alpha_3}}$$

Integrating the last inequality from t_1 to t to get

$$\omega_3(t)-\omega_3(t_1)\geq\int_{t_1}^{t}\left(\frac{1}{\zeta_3(s)}\int_{s}^{\delta(s)}q_3(s)\,y_1^{\alpha_3}(\sigma_3(s))dv\right)^{\frac{1}{\alpha_3}}ds \tag{17}$$

We claim that $\liminf_{t\to\infty}y_1(t)=0$, otherwise $\liminf_{t\to\infty}y_1(t)=l_1>0$, thus there exist $t_2\geq t_1$ large enough such that $y_1(t)\geq l_1$ hence

$$\omega_3(t)-\omega_3(t_1)\geq l_1\int_{t_2}^{t}\left(\frac{1}{\zeta_3(s)}\int_{s}^{\delta(s)}q_3(s)\,dv\right)^{\frac{1}{\alpha_3}}ds$$

Letting $t\to\infty$ with the virtue of condition (13), from the last inequality we obtain $\lim_{t\to\infty}\omega_3(t)=\infty$, which is a contradiction since $\omega_3(t)<0$ and bounded. Thus $\liminf_{t\to\infty}y_1(t)=0$, so there is a sequence $t_n\to\infty$, as $n\to\infty$, such that $\lim_{n\to\infty}y_1(t_n)=0$, we claim that $\lim_{t\to\infty}\omega_1(t)=L_1=0$, otherwise if $L_1>0$ then from (14a) it follows

$$y_1(t_n)\geq\omega_1(t_n)-\mathcal{P}_1(t_n)\omega_1(\tau_1(t_n))\geq\omega_1(t_n)-\mu\omega_1(\tau_1(t_n))$$

As $n\to\infty$, it follows from the last inequality $0\geq(1-\mu)L_1$ a contradiction, and so $\lim_{t\to\infty}\omega_1(t)=0$. Implies to $\lim_{t\to\infty}y_1(t)=0$. Hence $y_2(t),y_3(t)$ are oscillatory while $y_1(t)$, either oscillatory or nonoscillatory tends to zero as $t\to\infty$.

Case 3. Suppose that $Y(t)\in \boldsymbol{L}_8$. Then we have

$$\omega_1(t)=y_1(t)+\mathcal{P}_1(t)y_1(\tau_1(t))\Longrightarrow\omega_1(t)\leq y_1(t)<0\,,$$
$$\omega_2(t)=y_2(t)+\mathcal{P}_2(t)y_2(\tau_2(t))\Longrightarrow\omega_2(t)\leq y_2(t)<0,$$
$$\omega_3(t)=y_3(t)+\mathcal{P}_3(t)y_3(\tau_3(t))\Longrightarrow\omega_3(t)\geq y_3(t)>0.$$

Then

$$\begin{aligned}
&y_1(\sigma_1(t))\leq\left(1-\mathcal{P}_1(\sigma_1(t))\right)\omega_1(\tau_1(\sigma_1(t)) && (a)\\
&y_2(\sigma_2(t))\leq\left(1-\mathcal{P}_2(\sigma_2(t))\right)\omega_2(\tau_2(\sigma_2(t)) && (b)\\
&y_3(t)\geq\omega_3(t)-\mathcal{P}_3(t)\omega_3(\tau_3(t)) && (c)
\end{aligned} \tag{18}$$

Integrating the first equation of system (1) from t to $\delta(t)$ and using (18b) to get

$$\zeta_1(\delta(t))(\omega_1'(\delta(t)))^{\alpha_1}-\zeta_1(t)(\omega_1'(t))^{\alpha_1}=-\int_{t}^{\delta(t)}q_1(s)y_2^{\alpha_1}(\sigma_2(s))\,ds$$

$$\zeta_1(t)(\omega_1'(t))^{\alpha_1}\leq\int_{t}^{\delta(t)}q_1(s)\left(1-\mathcal{P}_2(\sigma_2(s))\right)^{\alpha_1}\omega_2^{\alpha_1}(\tau_2(\sigma_2(s))ds,$$

$$\omega_1'(t)\leq\omega_2(\tau_2(\sigma_2(t))\left(\frac{1}{\zeta_1(t)}\int_{t}^{\delta(t)}q_1(s)\left(1-\mathcal{P}_2(\sigma_2(s))\right)^{\alpha_1}ds\right)^{\frac{1}{\alpha_1}}$$

Integrating the last inequality from t_1 to t to get

$$\omega_1(t)-\omega_1(t_1)\leq\omega_2(\tau_1(\sigma_2(t_1))\int_{t_1}^{t}\left(\frac{1}{\zeta_1(s)}\int_{s}^{\delta(s)}q_1(s)\left(1-\mathcal{P}_2(\sigma_2(v))\right)^{\alpha_1}dv\right)^{\frac{1}{\alpha_1}}ds$$

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Letting $t \to \infty$ and in view of condition (13), the last inequality leads to $\lim_{t\to\infty} \omega_1(t) = -\infty$, which is a contradiction with boundednees of $\omega_1(t)$.

Integrating the second equation of system (1) from t to $\delta(t)$ and using (18c) to get

$$\zeta_2(\delta(t))(\omega_2'(\delta(t)))^{\alpha_2} - \zeta_2(t)(\omega_2'(t))^{\alpha_2} = -\int_t^{\delta(t)} q_2(s) y_3^{\alpha_2}(\sigma_2(s))\, ds$$

$$\zeta_2(t)(\omega_2'(t))^{\alpha_2} \geq \int_t^{\delta(t)} q_2(s) y_3^{\alpha_2}(\sigma_2(s))\, ds$$

We claim that $\liminf_{t\to\infty} y_3(t) = 0$, otherwise $\liminf_{t\to\infty} y_3(t) = l_3 > 0$, thus there exist $t_2 \geq t_1$ large enough such that $y_3(t) \geq l_3$, hence

$$\omega_2(t) - \omega_2(t_1) \geq l_3 \int_{t_2}^{t} \left(\frac{1}{\zeta_2(s)} \int_s^{\delta(s)} q_2(s)\, dv \right)^{\frac{1}{\alpha_2}} ds$$

Letting $t \to \infty$ with the virtue of condition (13), from the last inequality we obtain $\lim_{t\to\infty} \omega_2(t) = \infty$, which is a contradiction since $\omega_2(t) < 0$ and bounded. Thus $\liminf_{t\to\infty} y_3(t) = 0$, so there is a sequence $t_n \to \infty$, as $n \to \infty$, such that $\lim_{n\to\infty} y_3(t_n) = 0$, we claim that $\lim_{t\to\infty} \omega_3(t) = L_3 = 0$, otherwise if $L_3 > 0$ then from (18c) it follows

$$y_3(t_n) \geq \omega_3(t_n) - \mathcal{P}_3(t_n)\omega_3(\tau_3(t_n)) \geq \omega_3(t_n) - \mu\omega_3(\tau_3(t_n))$$

As $n \to \infty$, it follows from the last inequality $0 \geq (1-\mu)L_3$ a contradiction, and so $\lim_{t\to\infty} \omega_3(t) = 0$, implies to $\lim_{t\to\infty} y_3(t) = 0$.

Integrating the third equation of system (1) from t to $\delta(t)$ to get

$$\zeta_3(\delta(t))(\omega_3'(\delta(t)))^{\alpha_3} - \zeta_3(t)(\omega_3'(t))^{\alpha_3} = -\int_t^{\delta(t)} q_3(s) y_1^{\alpha_3}(\sigma_1(s))\, ds$$

$$\zeta_3(t)(\omega_3'(t))^{\alpha_3} \leq \int_t^{\delta(t)} q_3(s) y_1^{\alpha_3}(\sigma_3(s))\, ds$$

$$\omega_3'(t) \leq \omega_1(\tau_1(\sigma_1(t))) \left(\frac{1}{\zeta_3(t)} \int_t^{\delta(t)} q_3(s) \left(1 - \mathcal{P}_3(\sigma_3(s))\right)^{\alpha_3} ds \right)^{\frac{1}{\alpha_3}}$$

Integrating the last inequality from t_1 to t to get

$$\omega_3(t) - \omega_3(t_1) \leq \omega_1(\tau_1(\sigma_1(t_1))) \int_{t_1}^{t} \left(\frac{1}{\zeta_3(s)} \int_s^{\delta(s)} q_3(s) \left(1 - \mathcal{P}_3(\sigma_3(v))\right)^{\alpha_1} dv \right)^{\frac{1}{\alpha_1}} ds \quad (19)$$

Letting $t \to \infty$ and in view of condition (13), the last inequality leads to $\lim_{t\to\infty} \omega_3(t) = -\infty$, which is a contradiction. Hence $y_1(t), y_2(t)$ are oscillatory while $y_3(t)$, either oscillatory or nonoscillatory tends to zero as $t \to \infty$. Other cases can be handled in the same way. The proof is complete.

Corollary 4.1 Suppose that $\lambda = -1$ and (3), (13) are held. Then every solution of system (1) is either oscillatory or $\lim_{t\to\infty}|y_1(t)| = \lim_{t\to\infty}|y_2(t)| = \lim_{t\to\infty}|y_3(t)| = \infty$.

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Proof. Suppose that system (1) has a NOS $Y(t) = (y_1(t), y_2(t), y_3(t))^{\mathrm{T}}$ so by Lemma 3.1 Table 3, there are only the possible classes $\boldsymbol{L}_1 - \boldsymbol{L}_{13}$ to consider for $t \geq t_1 \geq t_0$. If $Y(t)$ is bounded then by Theorem 4.1, it follows that $Y(t)$ is oscillatory or nonoscillatory and converges to infinity as $t \to \infty$. Otherwise, $Y(t)$ is unbounded: then the following cases will be achieved:

Case 1. Suppose that $Y(t) \in \boldsymbol{L}_2$. By Lemma 3.1, it follows $\lim_{t\to\infty} y_3(t) = \infty$ and $\lim_{t\to\infty} y_2(t) = -\infty$. By integrating the first equation of system (1) from t to $\delta(t)$ and using (18b), this yields:

$$\zeta_1(\delta(t))(\omega_1'(\delta(t)))^{\alpha_1} - \zeta_1(t)(\omega_1'(t))^{\alpha_1} = -\int_t^{\delta(t)} q_1(s) y_2^{\alpha_1}(\sigma_2(s))\, ds$$

$$\zeta_1(t)(\omega_1'(t))^{\alpha_1} \leq \int_t^{\delta(t)} q_1(s) y_2^{\alpha_1}(\sigma_2(s))\, ds$$

$$\omega_1'(t) \leq \omega_2(\tau_2(\sigma_2(t)))\left(\frac{1}{\zeta_1(t)}\int_t^{\delta(t)} q_1(s)\left(1 - \mathcal{P}_2(\sigma_2(s))\right)^{\alpha_1} ds\right)^{\frac{1}{\alpha_1}}$$

Integrating the last inequality from t_1 to t to get

$$\omega_1(t) - \omega_1(t_1) \leq \omega_2(\tau_1(\sigma_2(t_1)))\int_{t_1}^{t}\left(\frac{1}{\zeta_1(s)}\int_s^{\delta(s)} q_1(s)\left(1 - \mathcal{P}_2(\sigma_2(v))\right)^{\alpha_1} dv\right)^{\frac{1}{\alpha_1}} ds$$

Letting $t \to \infty$ and in view of condition (13), the last inequality leads to $\lim_{t\to\infty} \omega_1(t) = -\infty$, which is a contradiction with positivity of $\omega_1(t)$. this leads to the solutions $Y(t)$ oscillates.

Case 2. Suppose that $Y(t) \in \boldsymbol{L}_4$. By Lemma 3.1, it follows $\lim_{t\to\infty} y_1(t) = \infty$ and $\lim_{t\to\infty} y_3(t) = -\infty$. Integrating the second equation of system (1) from t to $\delta(t)$ and using (18c) to get

$$\zeta_2(\delta(t))(\omega_2'(\delta(t)))^{\alpha_2} - \zeta_2(t)(\omega_2'(t))^{\alpha_2} = -\int_t^{\delta(t)} q_2(s) y_3^{\alpha_2}(\sigma_2(s))\, ds$$

$$\zeta_2(t)(\omega_2'(t))^{\alpha_2} \leq \int_t^{\delta(t)} q_2(s) y_3^{\alpha_2}(\sigma_2(s))\, ds,$$

$$\omega_2'(t) \leq \omega_3(\tau_3(\sigma_3(t)))\left(\frac{1}{\zeta_2(t)}\int_t^{\delta(t)} q_2(s)\left(1 - \mathcal{P}_3(\sigma_2(s))\right)^{\alpha_2} ds\right)^{\frac{1}{\alpha_2}}$$

Integrating the last inequality from t_1 to t to get

$$\omega_2(t) - \omega_2(t_1) \leq \omega_3\left(\tau_3(\sigma_3(t_1))\right)\int_{t_1}^{t}\left(\frac{1}{\zeta_2(s)}\int_s^{\delta(s)} q_2(s)\left(1 - \mathcal{P}_3(\sigma_2(v))\right)^{\alpha_2} dv\right)^{\frac{1}{\alpha_2}} ds \quad (20)$$

Letting $t \to \infty$ and by using the condition (15) leads to $\lim_{t\to\infty} \omega_2(t) = -\infty$, implies that $\lim_{t\to\infty} y_2(t) = -\infty$. We get $\lim_{t\to\infty}|y_1(t)| = \lim_{t\to\infty}|y_2(t)| = \lim_{t\to\infty}|y_3(t)| = \infty$. The proof of other cases are similar to the proof the cases **1 or 2.**

Theorem 4.2 Assume that $0 \leq \mathcal{P}_i(t) \leq \mathcal{P}_i < 1, \lambda = 1$. Let y_1, y_2, y_3 are a NOS of (1) and suppose the corresponding $(\omega_1, \omega_2, \omega_3) \in K_2$ satisfies. If

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$$\begin{cases} \lim_{t\to\infty} \sup \int_T^t \left[\frac{1}{\zeta_1(v)} \int_v^\infty q_1(s)ds\right]^{\frac{1}{\alpha_1}} dv = \infty \\ \lim_{t\to\infty} \sup \int_T^t \left[\frac{1}{\zeta_2(v)} \int_v^\infty q_2(s)ds\right]^{\frac{1}{\alpha_2}} dv = \infty , \quad t \geq T \\ \lim_{t\to\infty} \sup \int_T^t \left[\frac{1}{\zeta_3(v)} \int_v^\infty q_3(s)ds\right]^{\frac{1}{\alpha_3}} dv = \infty \end{cases} \tag{21}$$

Then

$$\begin{cases} \lim_{t\to\infty} y_1 = \lim_{t\to\infty} \omega_1 = 0, \\ \lim_{t\to\infty} y_2 = \lim_{t\to\infty} \omega_2 = 0, \\ \lim_{t\to\infty} y_3 = \lim_{t\to\infty} \omega_3 = 0. \end{cases}$$

Proof. Suppose that y_1, y_2, y_3 are positive solution of (1). since $\omega_2(t) > 0$ and $\omega_2'(t) < 0$, then there exists finite $\hbar_2$ such that

$$\lim_{t\to\infty} \omega_2(t) = \hbar_2.$$

We shall prove that $\hbar_2 = 0$. Assume that $\hbar_2 > 0$. Then fore any $\varepsilon > 0$, we have $\hbar_2 < \omega_2(t) < \hbar_2 + \varepsilon$, eventually. Choose $0 < \varepsilon < \frac{\hbar_2(1-p_2)}{p_2}$. It is easy to verify that

$y_2(t) = \omega_2(t) - p_2 y_2(\tau_2(t)) > \hbar_2 - p_2(\hbar_2 + \varepsilon) = k_2(\hbar_2 + \varepsilon) > k_2\omega_2(t)$,

Where $k_2 = \frac{\hbar_2 - p_2(\hbar_2+\varepsilon)}{\hbar_2+\varepsilon} > 0$. By similar way, we have $k_1 > 0, k_3 > 0$.

Using the above inequality, we obtain from (1)

$$\begin{cases} \left(\zeta_1(t)\left(\omega_1'(t)\right)^{\alpha_1}\right)' \geq k_2^{\alpha_1} q_1(t)\omega_2^{\alpha_1}(\sigma_1(t)) \\ \left(\zeta_2(t)\left(\omega_2'(t)\right)^{\alpha_2}\right)' \geq k_3^{\alpha_2} q_2(t)\omega_3^{\alpha_2}(\sigma_2(t)), \\ \left(\zeta_3(t)\left(\omega_3'(t)\right)^{\alpha_3}\right)' \geq k_1^{\alpha_3} q_3(t)\omega_1^{\alpha_3}(\sigma_3(t)) \end{cases} \tag{22}$$

Integrating the first inequality of (22) from t to ∞, we get

$$\zeta_1(t)\left(\omega_1'(t)\right)^{\alpha_1} \leq -k_2^{\alpha_1} \int_t^\infty q_1(t)\omega_2^{\alpha_1}(\sigma_1(t))\, ds$$

Using $\omega_2(\sigma_1(t)) \geq \hbar_2$, we see that

$$\omega_1'(t) \leq -k_2\hbar_2 \left[\frac{1}{\zeta_1(t)} \int_t^\infty q_1(s)ds\right]^{\frac{1}{\alpha_1}},$$

Integrating from t_1 to t, we obtain

$$\omega_1(t) - \omega_1(t_1) \leq -k_2\hbar_2 \int_{t_1}^t \left[\frac{1}{\zeta_1(u)} \int_u^\infty q_1(s)ds\right]^{\frac{1}{\alpha_1}} du, \tag{23}$$

As $t \longrightarrow \infty$, a contradiction will be occurred. Therefore, $\hbar_2 = 0$. Moreover, the inequality $0 \leq y_2(t) \leq \omega_2(t)$ implies $\lim_{t\to\infty} y_2(t) = 0$.

By the same way we can proof that $\lim_{t\to\infty} y_3(t) = 0$ and $\lim_{t\to\infty} y_1(t) = 0$ and the proof is complete.

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Theorem 4.3 Assume that $\boldsymbol{\lambda} = \mathbf{1}$, and the conditions (3), (13) holds. Then all the solutions $Y(t)$ of the system (1) belong to the classes $\boldsymbol{K}_{10}, \boldsymbol{K}_{12}, \boldsymbol{K}_{14}$ are oscillate.

Proof. Suppose that system (1) has a NOS $Y(t)$ so by Lemma 2.2, table 1, there are only the possible cases $\boldsymbol{K}_{10}, \boldsymbol{K}_{12}, \boldsymbol{K}_{14}$, to consider for $t \geq t_1 \geq t_0$:

***Case* 1**. Suppose that $Y(t) \in \boldsymbol{K}_{10}$. Then we have

$$\begin{aligned} \omega_1(t) &= y_1(t) + \mathcal{P}_1(t) y_1\big(\tau_1(t)\big) \Longrightarrow \omega_1(t) \geq y_1(t) \\ \omega_2(t) &= y_2(t) + \mathcal{P}_2(t) y_2\big(\tau_2(t)\big) \Longrightarrow \omega_2(t) \leq y_2(t) \\ \omega_3(t) &= y_3(t) + \mathcal{P}_3(t) y_3\big(\tau_3(t)\big) \Longrightarrow \omega_3(t) \leq y_3(t) \end{aligned} \tag{24}$$

Then

$$\begin{aligned} y_1(\sigma_3(t)) &\geq \Big(1 - \mathcal{P}_1\big(\sigma_3(t)\big)\Big)\, \omega_1\big(\sigma_3(t)\big) \\ y_2(\sigma_1(t)) &\leq \Big(1 - \mathcal{P}_2\big(\sigma_1(t)\big)\Big)\, \omega_2\big(\sigma_1(t)\big) \\ y_3(\sigma_2(t)) &\leq \Big(1 - \mathcal{P}_3\big(\sigma_2(t)\big)\Big)\, \omega_3\big(\sigma_2(t)\big) \end{aligned} \tag{25}$$

Integrating the first equation of system (1) from t to $\delta(t)$ to get

$$\zeta_1\big(\delta(t)\big)(\omega_1'\big(\delta(t)\big))^{\alpha_1} - \zeta_1(t)(\omega_1'(t))^{\alpha_1} = \int_t^{\delta(t)} q_1(s)\, y_2^{\alpha_1}\big(\sigma_1(s)\big)\, ds$$

$$-\zeta_1(t)(\omega_1'(t))^{\alpha_1} \leq \int_t^{\delta(t)} q_1(s) y_2^{\alpha_1}\big(\sigma_1(s)\big)\, ds,$$

$$-\omega_1'(t) \leq \left(\frac{\omega_2^{\alpha_1}\big(\sigma_1(t)\big)}{\zeta_1(t)} \int_t^{\delta(t)} q_1(s) \Big(1 - \mathcal{P}_2\big(\sigma_1(s)\big)\Big)^{\alpha_1} ds\right)^{\frac{1}{\alpha_1}}$$

Integrating the last inequality from t_1 to t to get

$$-\omega_1(t) + \omega_1(t_1) \leq \omega_2\big(\sigma_1(t_1)\big) \int_{t_1}^{t} \left(\frac{1}{\zeta_1(s)}\right)^{\frac{1}{\alpha_1}} \left(\int_s^{\delta(s)} q_1(s) \Big(1 - \mathcal{P}_2\big(\sigma_1(v)\big)\Big)^{\alpha_1} dv\right)^{\frac{1}{\alpha_1}} ds$$

Letting $t \to \infty$ and by using the condition (13) leads to $\lim_{t\to\infty} \omega_1(t) = \infty$, implies that $\lim_{t\to\infty} y_1(t) = \infty$, a contradiction.

Integrating the second equation of system (1) from t to $\delta(t)$ to get

$$\zeta_2\big(\delta(t)\big)(\omega_2'\big(\delta(t)\big))^{\alpha_2} - \zeta_2(t)(\omega_2'(t))^{\alpha_2} = \int_t^{\delta(t)} q_2(s) y_3^{\alpha_2}\big(\sigma_2(s)\big)\, ds,$$

$$\zeta_2\big(\delta(t)\big)(\omega_2'\big(\delta(t)\big))^{\alpha_2} - \zeta_2(t)(\omega_2'(t))^{\alpha_2} \leq \int_t^{\delta(t)} q_2(s) \Big(1 - \mathcal{P}_3\big(\sigma_2(s)\big)\Big)^{\alpha_2} \omega_3^{\alpha_2}\big(\sigma_2(s)\big) ds,$$

$$-\zeta_2(t)(\omega_2'(t))^{\alpha_2} \leq \int_t^{\delta(t)} q_2(s) \Big(1 - \mathcal{P}_3\big(\sigma_2(s)\big)\Big)^{\alpha_2} \omega_3^{\alpha_2}\big(\sigma_2(s)\big) ds,$$

$$-\omega_2'(t) \leq \left(\frac{\omega_3^{\alpha_2}\Big(\sigma_2\big(\delta(t)\big)\Big)}{\zeta_2(t)} \int_t^{\delta(t)} q_2(s) \Big(1 - \mathcal{P}_3\big(\sigma_2(s)\big)\Big)^{\alpha_2} ds\right)^{\frac{1}{\alpha_1}}$$

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Integrating the last inequality from t_1 to t to get

$$-\omega_2(t)+\omega_2(t_1)\le\int_{t_1}^{t}\left(\frac{\omega_3^{\alpha_2}\left(\sigma_2(\delta(s))\right)}{\zeta_2(s)}\int_{s}^{\delta(s)}q_2(s)\left(1-\mathcal{P}_3(\sigma_2(v))\right)^{\alpha_2}dv\right)^{\frac{1}{\alpha_2}}ds$$

$$-\omega_2(t)+\omega_1(t_2)\le\omega_3\left(\sigma_2(\delta(t))\right)\int_{t_1}^{t}\left(\frac{1}{\zeta_2(s)}\right)^{\frac{1}{\alpha_1}}\left(\int_{s}^{\delta(s)}q_2(s)\left(1-\mathcal{P}_3(\sigma_2(v))\right)^{\alpha_2}dv\right)ds$$

Letting $t\to\infty$ and by using the condition (13) leads to $\lim_{t\to\infty}\omega_2(t)=\infty$, implies that $\lim_{t\to\infty}y_2(t)=\infty$, a contradiction.

Integrating the third equation of system (1) from t to $\delta(t)$ to get

$$\zeta_3(\delta(t))(\omega_3'(\delta(t)))^{\alpha_3}-\zeta_3(t)(\omega_3'(t))^{\alpha_3}=\int_{t}^{\delta(t)}q_3(s)y_1^{\alpha_3}(\sigma_3(s))\,ds$$

$$-\zeta_3(t)(\omega_3'(t))^{\alpha_3}\ge\int_{t}^{\delta(t)}q_3(s)y_1^{\alpha_3}(\sigma_3(s))\,ds,$$

$$-\omega_3'(t)\ge\left(\frac{\omega_1^{\alpha_3}(\sigma_3(t))}{\zeta_3(t)}\int_{t}^{\delta(t)}q_3(s)\left(1-\mathcal{P}_1(\sigma_3(s))\right)^{\alpha_3}ds\right)^{\frac{1}{\alpha_1}}$$

Integrating the last inequality from t_1 to t to get

$$-\omega_3(t)+\omega_3(t_1)\ge\int_{t_1}^{t}\left(\frac{\omega_1^{\alpha_3}(\sigma_3(s))}{\zeta_3(s)}\int_{t}^{\delta(t)}q_3(v)\left(1-\mathcal{P}_1(\sigma_3(v))\right)^{\alpha_3}dv\right)^{\frac{1}{\alpha_2}}ds$$

$$-\omega_3(t)+\omega_3(t_2)\ge\omega_1(\sigma_3(t_1))\int_{t_1}^{t}\left(\frac{1}{\zeta_2(s)}\int_{s}^{\delta(s)}q_2(s)\left(1-\mathcal{P}_3(\sigma_2(v))\right)^{\alpha_2}dv\right)^{\frac{1}{\alpha_2}}ds$$

Letting $t\to\infty$ and by using the condition (13) leads to $\lim_{t\to\infty}\omega_3(t)=-\infty$, implies that $\lim_{t\to\infty}y_3(t)=-\infty$, a contradiction. This leads to the solutions $Y(t)$ oscillates. Other cases can be handled in the same way. The proof is complete.

Corollary 4.2 Suppose that (3) and (13) are held. Then every solution of the system (1) with $\lambda=1$ is either oscillatory, almost oscillatory, $\lim_{t\to\infty}y_i(t)=0$ or $\lim_{t\to\infty}|y_i(t)|=\infty,\ i=1,2,3.$

Proof. Suppose that system (1) has a nonoscillatory solution $Y(t)$, so by Lemma 2.2, table 1, there are only the possible classes $\boldsymbol{K}_1-\boldsymbol{K}_{14}$ to consider for $t\ge t_1\ge t_0$. If $Y(t)$ is bounded then by Theorem 4.3, it follows that $Y(t)$ is oscillatory. Otherwise, $Y(t)$ is unbounded:

Case 1. Suppose that $Y(t)\in\boldsymbol{K}_1$. By Lemma 2.2, it follows $\lim_{t\to\infty}\omega_i(t)=\infty$, it follows that $\lim_{t\to\infty}y_i(t)=\infty\quad i=1,2,3.$

Case 2. Suppose that $Y(t)\in\boldsymbol{K}_{10},\boldsymbol{K}_{12},\boldsymbol{K}_{14}$. By Theorem 4.3, $\lim_{t\to\infty}|y_i(t)|=\infty,\ i=1,2,3.$

Case 3. Suppose that $Y(t)\in\boldsymbol{K}_3$. By Lemma 2.1, it follows $\lim_{t\to\infty}y_3(t)=\infty,$

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Then we have

$$\omega_1(t) = y_1(t) + \mathcal{P}_1(t)y_1(\tau_1(t)) \Longrightarrow \omega_1(t) \geq y_1(t)$$
$$\omega_2(t) = y_2(t) + \mathcal{P}_2(t)y_2(\tau_2(t)) \Longrightarrow \omega_2(t) \geq y_2(t)$$
$$\omega_3(t) = y_3(t) + \mathcal{P}_3(t)y_3(\tau_3(t)) \Longrightarrow \omega_3(t) \geq y_3(t)$$

Then

$$\begin{aligned} y_1(t) &\geq \omega_1(t) - \mathcal{P}_1(t)\omega_1(\tau_1(t)) && (a) \\ y_2(t) &\geq \omega_2(t) - \mathcal{P}_2(t)\omega_2(\tau_2(t)) && (b) \\ y_3(\sigma_2(t)) &\geq \left(1 - \mathcal{P}_3(\sigma_2(t))\right)\omega_3(\sigma_2(t)) && (c) \end{aligned} \tag{26}$$

Integrating the first equation of system (1) from t to $\delta(t)$ to get

$$\zeta_1(\delta(t))(\omega_1'(\delta(t)))^{\alpha_1} - \zeta_1(t)(\omega_1'(t))^{\alpha_1} = \int_t^{\delta(t)} q_1(s)y_2^{\alpha_1}(\sigma_1(s))\,ds$$

$$-\zeta_1(t)(\omega_1'(t))^{\alpha_1} \geq \int_t^{\delta(t)} q_1(s)y_2^{\alpha_1}(\sigma_1(s))\,ds,$$

We claim that $\liminf_{t\to\infty} y_2(t) = 0$, otherwise $\liminf_{t\to\infty} y_2(t) = l_2 > 0$, thus there exist $t_2 \geq t_1$ large enough such that $y_2(t) \geq l_2$ hence

Integrating the last inequality from t_2 to t we get

$$-\omega_1(t) + \omega_1(t_1) \geq l_2 \int_{t_2}^{t} \left(\frac{1}{\zeta_1(s)} \int_s^{\delta(s)} q_1(s)\,dv\right)^{\frac{1}{\alpha_1}} ds$$

Letting $t \to \infty$ with the virtue of condition (21), from the last inequality we obtain $\lim_{t\to\infty} \omega_1(t) = -\infty \Longrightarrow \lim_{t\to\infty} y_1(t) = -\infty$ which is a contradiction since $y_1(t) > 0$ and bounded.

Thus, $\liminf_{t\to\infty} y_2(t) = 0$, so there is a sequence $t_n \to \infty$, as $n \to \infty$, such that $\lim_{n\to\infty} y_2(t_n) = 0$, we claim that $\lim_{t\to\infty} \omega_2(t) = L_2 = 0$, otherwise if $L_2 > 0$ then from (b)-(34) it follows

$$y_2(t_n) \geq \omega_2(t_n) - \mathcal{P}_2(t_n)\omega_2(\tau_2(t_n)) \geq \omega_2(t_n) - \mu\omega_2(\tau_2(t_n))$$

As $n \to \infty$, it follows from the last inequality $0 \geq (1-\mu)L_2$ a contradiction, and so $\lim_{t\to\infty} \omega_2(t) = 0$, implies to $\lim_{t\to\infty} y_2(t) = 0$.

Integrating the second equation of system (1) from t to $\delta(t)$ to get

$$\zeta_2(\delta(t))(\omega_2'(\delta(t)))^{\alpha_2} - \zeta_2(t)(\omega_2'(t))^{\alpha_2} = \int_t^{\delta(t)} q_2(s)y_3^{\alpha_2}(\sigma_2(s))\,ds$$

$$\zeta_2(\delta(t))(\omega_2'(\delta(t)))^{\alpha_2} - \zeta_2(t)(\omega_2'(t))^{\alpha_2} \geq \int_t^{\delta(t)} q_2(s)\left(1 - \mathcal{P}_3(\sigma_2(s))\right)^{\alpha_2} \omega_3^{\alpha_2}(\sigma_2(s))ds,$$

$$-\omega_2'(t) \geq \left(\frac{\omega_3^{\alpha_2}(\sigma_2(t))}{\zeta_2(t)} \int_t^{\delta(t)} q_2(s)\left(1 - \mathcal{P}_3(\sigma_2(s))\right)^{\alpha_2} ds\right)^{\frac{1}{\alpha_1}}$$

Integrating the last inequality from t_1 to t to get

$$-\omega_2(t)+\omega_2(t_1)\geq\int_{t_1}^{t}\left(\frac{\omega_3^{\alpha_2}(\sigma_2(s))}{\zeta_2(s)}\int_s^{\delta(s)}q_2(s)\left(1-\mathcal{P}_3(\sigma_2(v))\right)^{\alpha_2}dv\right)^{\frac{1}{\alpha_2}}ds$$

$$-\omega_2(t)+\omega_2(t_2)\geq\omega_3(\sigma_2(t_1))\int_{t_1}^{t}\left(\frac{1}{\zeta_2(s)}\right)^{\frac{1}{\alpha_1}}\left(\int_s^{\delta(s)}q_2(s)\left(1-\mathcal{P}_3(\sigma_2(v))\right)^{\alpha_2}dv\right)ds$$

Letting $t\to\infty$ and by using the condition (13) leads to $\lim_{t\to\infty}\omega_2(t)=-\infty$ then $\lim_{t\to\infty}y_2(t)=-\infty$. this is a contradiction with positivity of $\omega_2(t)$

Integrating the third equation of system (1) from t to $\delta(t)$ to get

$$\zeta_3(\delta(t))(\omega_3'(\delta(t)))^{\alpha_3}-\zeta_3(t)(\omega_3'(t))^{\alpha_3}=\int_t^{\delta(t)}q_3(s)y_1^{\alpha_3}(\sigma_3(s))\,ds$$

$$-\zeta_3(t)(\omega_3'(t))^{\alpha_3}\leq\int_t^{\delta(t)}q_3(s)y_1^{\alpha_3}(\sigma_3(s))\,ds,$$

We claim that $\liminf_{t\to\infty}y_1(t)=0$, otherwise $\liminf_{t\to\infty}y_1(t)=l_1>0$, thus there exist $t_2\geq t_1$ large enough such that $y_1(t)\geq l_1$ hence

Integrating the last inequality from t_2 to t we get

$$-\omega_3(t)+\omega_3(t_1)\leq l_1\int_{t_2}^{t}\left(\frac{1}{\zeta_3(s)}\int_s^{\delta(s)}q_3(s)\,dv\right)^{\frac{1}{\alpha_3}}ds$$

Letting $t\to\infty$ with the virtue of condition (13), from the last inequality we obtain $\lim_{t\to\infty}\omega_3(t)=-\infty$, which is a contradiction since $\omega_3(t)>0$ and bounded. Thus $\liminf_{t\to\infty}y_1(t)=0$, so there is a sequence $t_n\to\infty$, as $n\to\infty$, such that $\lim_{n\to\infty}y_1(t_n)=0$, we claim that $\lim_{t\to\infty}\omega_1(t)=L_1=0$, otherwise if $L_1>0$ then from (26a) it follows

$$y_1(t_n)\geq\omega_1(t_n)-\mathcal{P}_1(t_n)\omega_1(\tau_1(t_n))\geq\omega_1(t_n)-\mu\omega_1(\tau_1(t_n))$$

As $n\to\infty$, it follows from the last inequality $0\geq(1-\mu)L_1$ a contradiction, and so $\lim_{t\to\infty}\omega_1(t)=0$, Implies to $\lim_{t\to\infty}y_1(t)=0$.

This leads to the solutions $Y(t)=(y_1(t),y_2(t),y_3(t))^{\mathrm{T}}$ is almost oscillatory. Other cases can be handled in the same way**.** The proof is complete.

5. Examples

In this section, some illustrative examples are presented for the purpose of verifying the results that have been reached.

Example 5.1 Consider the NDS:

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$$\begin{cases}\left(\left(\left[\left(y_1(t)+\frac{1}{2}y_1(t-2)\right)'\right]^3\right)\right)'=3\left(1+\frac{1}{2e^2}\right)^3 e^3 y_2^3(t-1),\\ \left(\left[\left(y_2(t)+e^{-1}y_2(t-1)\right)'\right]^{\frac{3}{5}}\right)'=\frac{3}{5}\left(1+\frac{1}{e^2}\right)^{\frac{3}{5}} e^{\frac{9}{5}} y_3^{\frac{3}{5}}(t-3),\\ \left(\left[\left(y_3(t)+e^{-2}y_3(t-1)\right)'\right]^{\frac{1}{3}}\right)'=\frac{1}{3}\left(1+\frac{1}{3e^3}\right)^{\frac{1}{3}} e^{\frac{2}{3}} y_1^{\frac{1}{3}}(t-2)\end{cases} \tag{29}$$

Here

$\lambda=1,\alpha_1=3,\alpha_2=\frac{3}{5},\alpha_3=\frac{1}{3},\zeta_1(t)=\zeta_2(t)=\zeta_3(t)=1,$

$\mathcal{P}_1(t)=\frac{1}{2}\ ,\mathcal{P}_2(t)=e^{-1},\mathcal{P}_3(t)=e^{-2},$

$\tau_1(t)=t-2,\tau_2(t)=t-1,\tau_3(t)=t-1,\sigma_1(t)=t-1,\sigma_2(t)=t-3,\sigma_3(t)=t-2,$

$q_1(t)=3\left(1+\frac{1}{2e^2}\right)^3 e^3\ ,q_2(t)=\frac{3}{5}\left(1+\frac{1}{e^2}\right)^{\frac{3}{5}} e^{\frac{9}{5}},q_3(t)=\frac{1}{3}\left(1+\frac{1}{3e^3}\right)^{\frac{1}{3}} e^{\frac{2}{3}}.$

$\omega_1(t)=y_1(t)+\frac{1}{2}y_1(t-2)>0,\omega_1'(t)>0,\omega_1''(t)>0,$

$\omega_2(t)=y_2(t)+e^{-1}y_2(t-1)>0,\omega_2'(t)>0,\omega_2''(t)>0,$

$\omega_3(t)=y_3(t)+e^{-2}y_3(t-1)>0,\omega_3'(t)>0,\omega_3''(t)>0,$

Then $(e^t,e^t,e^t)\in K_1$.

And

$$\int_0^\infty\left(\frac{1}{\zeta_1(t)}\right)^{\frac{1}{\alpha_1}}dt=\int_0^\infty\left(\frac{1}{\zeta_2(t)}\right)^{\frac{1}{\alpha_2}}dt=\int_0^\infty\left(\frac{1}{\zeta_3(t)}\right)^{\frac{1}{\alpha_3}}dt=\int_0^\infty dt=\infty.$$

$$\limsup_{t\to\infty}\int_0^t\left(\int_s^{\delta(s)}3\left(1+\frac{1}{2e^2}\right)^3 e^3\left(1-e^{-1}\right)^3dv\right)^{\frac{1}{3}}ds=\infty.$$

$$\limsup_{t\to\infty}\int_0^t\left(\int_s^{\delta(s)}\frac{3}{5}\left(1+\frac{1}{e^2}\right)^{\frac{3}{5}} e^{\frac{9}{5}}\left(1-e^{-2}\right)^{\frac{3}{5}}dv\right)^{\frac{5}{3}}ds=\infty.$$

$$\limsup_{t\to\infty}\int_0^t\left(\int_s^{\delta(s)}\frac{1}{3}\left(1+\frac{1}{3e^3}\right)^{\frac{1}{3}} e^{\frac{2}{3}}\left(1-\frac{1}{2}\right)^{\frac{1}{3}}dv\right)^3ds=\infty.$$

all the conditions of Theorem 4.3, satisfied. The solution (e^t,e^t,e^t) is NOS tends to infinity as $t\to\infty$.

Example 5.2 Consider the NDS:

$$\begin{cases}\left(\left(\left[\left(y_1(t)+e^{-1}y_1(t-1)\right)'\right]^3\right)\right)'=\frac{3}{4e^6}y_2^3(t-2),\\ \left(\left[\left(y_2(t)+e^{-2}y_2(t-2)\right)'\right]^{\frac{3}{5}}\right)'=\frac{3}{5}\sqrt[5]{3}\ e^{\frac{-3}{5}}\ y_3^{\frac{3}{5}}(t-1),\\ \left(\left(y_3(t)+e^{-3}y_3(t-3)\right)'\right)'=6e^{-4}y_2(t-4)\end{cases} \tag{30}$$

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Here

$\lambda = 1, \alpha_1 = 3, \alpha_2 = \frac{3}{5}, \alpha_3 = 1, \zeta_1(t) = \zeta_2(t) = \zeta_3(t) = 1,$

$\mathcal{P}_1(t) = e^{-1}, \mathcal{P}_2(t) = e^{-2}, \mathcal{P}_3(t) = e^{-3}, q_1(t) = \frac{3}{4e^6}, q_2(t) = \frac{3}{5}\sqrt[5]{3}\, e^{\frac{-3}{5}}, q_3(t) = 6e^{-4}.$

$\tau_1(t) = t-1, \tau_2(t) = t-2, \tau_3(t) = t-3, \sigma_1(t) = t-2, \sigma_2(t) = t-1, \sigma_3(t) = t-4,$

$\omega_1(t) = y_1(t) + e^{-1}y_1(t-1) > 0, \omega_1'(t) < 0, \omega_1''(t) > 0,$

$\omega_2(t) = y_2(t) + e^{-2}y_2(t-2) > 0, \omega_2'(t) < 0, \omega_2''(t) > 0,$

$\omega_3(t) = y_3(t) + e^{-3}y_3(t-3) > 0, \omega_3'(t) < 0, \omega_3''(t) > 0,$

Then $(e^{-t}, 2e^{-t}, 3e^{-t}) \in K_2$

$$\int_0^\infty \left(\frac{1}{\zeta_1(t)}\right)^{\frac{1}{\alpha_1}} dt = \int_0^\infty \left(\frac{1}{\zeta_2(t)}\right)^{\frac{1}{\alpha_2}} dt = \int_0^\infty \left(\frac{1}{\zeta_3(t)}\right)^{\frac{1}{\alpha_3}} dt = \int_0^\infty dt = \infty,$$

$$\limsup_{t\to\infty} \int_0^t \left(\int_s^\infty \frac{3}{4e^6}\, dv\right)^{\frac{1}{3}} ds = \infty, \limsup_{t\to\infty} \int_0^t \left(\int_s^\infty \frac{3}{5}\sqrt[5]{3}\, e^{\frac{-3}{5}}\, dv\right)^{\frac{5}{3}} ds = \infty,$$

$$\limsup_{t\to\infty} \int_0^t \int_s^\infty 6e^{-4}\, dv\, ds = \infty.$$

all the conditions of Theorem 4.2, satisfied. The solution $(e^{-t}, 2e^{-t}, 3e^{-t})$ is nonoscillatory solution tends to zero as $t \to \infty$.□

Example 5.3 Consider the NDS.

$$\begin{cases} \left(a\left(\left(x(t) + \frac{1}{4}x(t-2\pi)\right)''\right)^3\right)' = -3a\left(\frac{5}{4}\right)^3 \sin^2 t\, y\left(t - \frac{3\pi}{2}\right) \\ \left(b\left(\left(y(t) + \frac{1}{4}y(t-2\pi)\right)''\right)^3\right)' = -3b\left(\frac{5}{4}\right)^3 \sin^2 t\, x\left(t - \frac{3\pi}{2}\right) \\ \left(c\left(\left(y(t) + \frac{1}{4}y(t-2\pi)\right)''\right)^3\right)' = -3c\left(\frac{5}{4}\right)^3 \sin^2 t\, x\left(t - \frac{3\pi}{2}\right) \end{cases} \quad (31)$$

From (31) we have

$\zeta_1(t) = a, \zeta_2(t) = b, \zeta_3(t) = c, \mathcal{P}_1(t) = \mathcal{P}_2(t) = \mathcal{P}_3(t) = \frac{1}{4}, \alpha_1 = \alpha_2 = \alpha_3 = 3$

$\sigma_1(t) = \sigma_2(t) = \sigma_3(t) = t - \frac{3\pi}{2}, \tau_1(t) = \tau_2(t) = \tau_3(t) = t - 2\pi.$

$q_1(t) = -3a\left(\frac{5}{4}\right)^3 \sin^2 t, q_2(t) = -3b\left(\frac{5}{4}\right)^3 \sin^2 t, q_3(t) = -3c\left(\frac{5}{4}\right)^3 \sin^2 t,$

to test the conditions (5) and (15) note that $\delta(t) - t \geq c_1 > 0, t \geq T$ for some $c_1 > 0$ then

$$\lim_{t\to\infty} \int_T^t \int_s^{\delta(s)} \left(\frac{1}{\zeta_1(\xi)}\right)^{\frac{1}{\alpha}} d\xi\, ds = \lim_{t\to\infty} \int_T^t \int_s^{\delta(s)} \left(\frac{1}{a}\right)^{\frac{1}{3}} d\xi\, ds = \left(\frac{1}{a}\right)^{\frac{1}{3}} \lim_{t\to\infty} \int_T^t \int_s^{\delta(s)} d\xi\, ds = \infty$$

By the same way

$$\lim_{t\to\infty} \int_T^t \int_s^{\delta(s)} \left(\frac{1}{\zeta_2(\xi)}\right)^{\frac{1}{\alpha_2}} d\xi\, ds = \infty, \lim_{t\to\infty} \int_T^t \int_s^{\delta(s)} \left(\frac{1}{\zeta_3(\xi)}\right)^{\frac{1}{\alpha_3}} d\xi\, ds = \infty$$

Now

$$\lim_{t\to\infty}\int_T^t\int_s^{\delta(s)}\left(\frac{1}{\zeta_1(\xi)}\int_\xi^{\delta(\xi)} q_1(\theta)\left(1-\mathcal{P}_2\big(\sigma_1(\theta)\big)\right)d\theta\right)^{\frac{1}{3}} d\xi ds =$$

$$\lim_{t\to\infty}\int_T^t\int_s^{\delta(s)}\left(\frac{1}{a}\int_\xi^{\delta(\xi)} 3a\left(\frac{5}{4}\right)^3 \sin^2(\theta)\left(1-\frac{1}{4}\right)d\theta\right)^{\frac{1}{3}} d\xi ds$$

$$=\left(\frac{3}{4}\right)^{\frac{1}{3}}\left(\frac{5}{4}\right)(3)^{\frac{1}{3}}\lim_{t\to\infty}\int_T^t\int_s^{\delta(s)}\left(\int_\xi^{\delta(\xi)}\sin^2\theta\, d\theta\right)^{\frac{1}{3}} d\xi ds$$

$$=\left(\frac{3}{4}\right)^{\frac{1}{3}}\left(\frac{5}{4}\right)(3)^{\frac{1}{3}}\lim_{t\to\infty}\int_T^t\int_s^{\delta(s)}\left(\frac{1}{2}\int_\xi^{\delta(\xi)}(1-\cos 2\theta)\, d\theta\right)^{\frac{1}{3}} d\xi ds$$

$$=\left(\frac{3}{4}\right)^{\frac{1}{3}}\left(\frac{5}{4}\right)(3)^{\frac{1}{3}}\left(\frac{1}{2}\right)^{\frac{1}{3}}\lim_{t\to\infty}\int_T^t\int_s^{\delta(s)}\left(\left[\theta-\frac{1}{2}\sin 2\theta\right]_\xi^{\delta(\xi)}\right)^{\frac{1}{3}} d\xi ds$$

$$\geq\left(\frac{3}{4}\right)^{\frac{1}{3}}\left(\frac{5}{4}\right)(3)^{\frac{1}{3}}\left(\frac{1}{2}\right)^{\frac{1}{3}}\lim_{t\to\infty}\int_T^t\int_s^{\delta(s)}(\delta(\xi)-\xi)^{\frac{1}{3}} d\xi ds \geq \left(\frac{3}{4}\right)^{\frac{1}{3}}\left(\frac{5}{4}\right)(3)^{\frac{1}{3}}\left(\frac{1}{2}\right)^{\frac{1}{3}} c_1^{\frac{4}{3}}\lim_{t\to\infty}(t-T)=\infty$$

Similarly,

$$\lim_{t\to\infty}\int_T^t\int_s^{\delta(s)}\left(\frac{1}{\zeta_2(\xi)}\int_\xi^{\delta(\xi)} q_2(\theta)\left(1-\mathcal{P}_3\big(\sigma_3(\theta)\big)\right)d\theta\right)^{\frac{1}{3}} d\xi ds=\infty$$

And

$$\lim_{t\to\infty}\int_T^t\int_s^{\delta(s)}\left(\frac{1}{\zeta_3(\xi)}\int_\xi^{\delta(\xi)} q_3(\theta)\left(1-\mathcal{P}_1\big(\sigma_1(\theta)\big)\right)d\theta\right)^{\frac{1}{3}} d\xi ds=\infty$$

so all conditions of theorem 4.1 hold, hence every bounded solution of (33) either oscillate or converge to zero as $t\to\infty$. For instance the solution $\big(y_1(t),y_2(t),y_3(t)\big)=(\sin t,\sin t,\sin t)$ is such an oscillatory solution.

Conclusion

This paper was developed to study the oscillation and asymptotic behavior solutions of system of second order half linear neutral differential equations. All nonoscillatory solution were classified into 16 different classes. Some sufficient conditions given to ensure most of these classes non-occurring. While from the rest classes it has been proven that under a certain condition all bounded solutions of this system are either oscillatory or converge to zero as $t\to\infty$. As well as all unbounded solution are either oscillates or tends to $\pm\infty$ as $t\to\infty$.

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Conflict of interests.

There are non-conflicts of interest.

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