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Some Concepts Related to Almost Injective(Projective)Semi modules

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بعض المفاهيم ذات العلاقة بشبه المقاسات الاغمارية (الاسقاطية) تقريبا

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ABSTRACT

In this research, the concepts injective and projective semi modules have been generalized to generalized-injective(projective) and essentially-injective semimodules, also the relationship of these concepts with almost-injective(projective) semimodules have been discussed. The semiring having the properties that every quasi- injective semimodule is injective and every almost self-injective semimodule is injective have been defined. Some results between these concepts have been obtained like, every almost-injective is generalized-injective semimodule, the definitions, where a semi module is indecomposable have been discussed. Dually for almost-projective semimodule. Also obtained that, the direct sum of two quasi-injective Ȑ-semimodule is quasi-injective when Ȑ is QI-semiring, this fact has been expanded to AQI-semiring.

Conclusion: In this paper, several generalization of injective and projective semi modules have been defined and some concepts with related to almost injective(projective)semimodules have been presented, also the concepts QI-semiring, AQI-semiring defined, some relationship between these concepts and almost injective(projective) semimodules discussed.

Keywords: Almost injective (projective) semimodules, generalized injective (projective) semimodules, Semimodules, Semirings.

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خلاصة

في هذا البحث ، تم تعميم مفاهيم الوحدات شبه الحَقْنِيَّة والإسقاطية على وحدات شبه الحقن المعممة (الإسقاطية) والحقنة بشكل أساسي، كما تمت مناقشة العلاقة بين هذه المفاهيم والنماذج شبه الحَقْنِيَّة (الإسقاطية). تم تحديد الصفات التي لها خصائص أن كل نصف وحدة شبه حقنة يتم حقنها ويتم تعريف كل شبه وحدة شبه ذاتية الحقن تقريبًا عن طريق الحقن. تم الحصول على بعض النتائج بين هذه المفاهيم مثل ، كل حقنة تقريبًا عبارة عن semimodule معمم بالحقن ، وقد تمت مناقشة التعريفات ، حيث لا يمكن تفكيك الوحدة النمطية. مزدوج للوحدة شبه الإسقاطية تقريبًا. تم الحصول أيضًا على أن المجموع المباشر لاثنين من شبه حقن شبه حلقية شبه حقنة عندما Ȑ هي QI−semiring ، وقد تم توسيع هذه الحقيقة إلى AQI−semiring.

الخلاصة: في هذا البحث ، تم تحديد عدة تعميمات للوحدات شبه الحقلية والإسقاطية ، كما تم تقديم بعض المفاهيم المتعلقة بنصف نماذج الحقن (الإسقاطي) تقريبًا ، وكذلك المفاهيم QI−semiring، AQI−semiring ، بعض العلاقة بين هذه المفاهيم وشبه الحقن (الإسقاطية) تقريبًا التي تمت مناقشتها.

الكلمات المفتاحية: شبه حقن (إسقاطية) ، شبه حقن معممة (إسقاطية)

1. Introduction and preliminaries.

Recall the definitions of "almost injective," "almost projective," "almost self-injective," and "almost self-projective" semimodules as in [1], [2], [3], and [4]. Some ideas pertaining to these concepts will be discussed in this research, as well as how they relate to one another. With some additional conditions appropriate to the case of modules, these concepts have also been defined for the class of semimodules based on the literature of modules. The terms "generalized - injective," "generalized - projective," "essentially - injective," and "length of semimodule" have been defined in this worksheet, and some of their relationships are discussed. If we consider two semimodules Ă and Ɓ, then Ă is known as almost Ɓ-injective when, for each subsemimodule Ұ of Ɓ and each homomorphism ξ: Ұ→ Ă , either there exists a homomorphism ζ:Ɓ→ Ă satisfying $\xi = \zeta i$, or there is a homomorphism γ: Ă →X such that $\gamma\xi = \pi i$ where $0 \neq X \leq_{\oplus}$ Ɓ, and π is the projection map[1]. A semimodule Ă is called almost injective semimodule if Ă is almost injective relative to every semimodule Ɓ[1]. A semimodule Ă is almost self-injective , if Ă is almost Ă-injective [3] and it is called almost Ɓ-projective, if for each surjective homomorphism α: Ɓ ⟶ Ӿ and every homomorphism δ:Ă⟶ Ӿ, either there is ψ: Ă ⟶ Ɓ where α $\psi = \delta$,or there is γ:Ұ⟶Ă where Ұ is nonzero summand of Ɓ such that $\delta\gamma = \alpha|_{Ұ}$, and Ă is called almost-projective if it is almost Ɓ-projective for every finitely generated semimodule Ɓ[2], Ă is said to be almost self-projective if it is almost Ă-projective[4]. In what follows Ȑ denotes a semiring with identity and Ă a unitary Ȑ-semimodule. A semiring Ȑ is a nonempty set with the binary operations addition and multiplication applied, satisfying (Ȑ, +) is a commutative monoid including an identity element 0 and (Ȑ, ×)) is a monoid including an identity 1. The addition is distributed by the multiplication, and 0 is the absorbing element[5]. A left Ȑ-semimodule Ă is a commutative monoid

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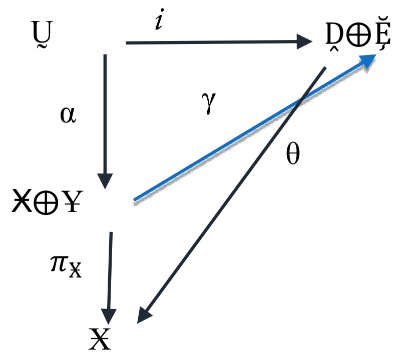
(Ă,+) together with operation Ȑ×Ă→Ă; defined by (ⱦ,a)⟼ⱦa such that ∀ⱦ, ɽ ∈Ȑ and a, b∈ Ă, ⱦ(a+ b)= ⱦa+ ⱦb; (ⱦ+ɽ)a= ⱦa+ ɽa;(ⱦɽ)a = ⱦ(ɽa); $0_{Ȑ}a = 0_{Ă}$ = ⱦ$0_{Ă}$ and $1_{Ȑ}a = a$ if there is $1_{Ȑ}$, Ă is called unitary semimodule (in the same way the right Ȑ-semimodule is defined)[5].

2. Some Properties of Almost Ɓ-injective and generalized Ɓ -Injective Semimodules

Definition2.1: A semimodule Ă is called generalized Ɓ- injective, if for any homomorphism α:Ṵ→Ă where Ṵ is sub-semimodule of Ɓ, there is decompositions Ɓ=Ḓ⊕Ḝ, Ă=Ӿ⊕Ұ, a homomorphism θ:Ḓ→Ӿ and monomorphism γ: Ұ →Ḝ such that Ṵ⊂Ḓ⊕γ(Ұ) with $\pi_{Ӿ}\alpha = \theta$ $\pi'_{Ḓ}|_{Ṵ} = \theta\, \pi'_{Ḓ} i_{Ṵ}$ and $\gamma\pi_{Ұ}\alpha = \pi'_{Ḝ}|_{Ṵ} = \pi'_{Ḝ}\, i_{Ṵ}$ and Ă is generalized self-injective semimodule if, Ă is generalized Ă-injective.



Proposition 2.2: If a semimodule Ă is generalized Ɓ-injective, hence Ă is an almost Ɓ-injective semimodule.

Proof. Suppose that Ṵ is a sub-semimodule of Ɓ and α:Ṵ→Ă be a homomorphism, so there is decompositions Ɓ=Ḓ⊕Ḝ, Ă=Ӿ⊕Ұ, a homomorphism θ:Ḓ→Ӿ and a monomorphism γ: Ұ →Ḝ such that Ṵ⊂ Ḓ⊕γ(Ұ) with $\pi_{Ӿ}\alpha = \theta\, \pi_{Ḓ}|_{Ṵ}$ and $\gamma\pi_{Ұ}\alpha = \pi_{Ḝ}|_{Ṵ} = \pi_{Ḝ}\, i_{Ṵ}$ .If α cannot be extended to Ɓ and Ḝ=0, then θ will be an extension of α to Ɓ, that is case1 in definition of almost Ɲ-injective holds. If Ḝ≠0, define δ: Ă→Ḓ by $\delta = \gamma\pi_{Ұ}$. For every u in Ṵ, $\delta\alpha$(u)= $\delta\alpha$(d+e)= $\delta(\theta$(d)+ θ'(e))= $\gamma\pi_{Ұ}(\theta$(d)+ θ'(e))= $\gamma(\theta'$(e))= e = $\pi_{Ḝ} i_{Ṵ}(u)$, θ':Ḝ→Ұ and $\pi_{Ḝ}$: Ɓ→Ḝ. Hence case two holds ■

Corollary 2.3: If a semimodule Ă is generalized self-injective, then it is an almost self-injective-semimodule.

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Remark 2.4:

(1)If Ɓ is indecomposable and Ă is generalized Ɓ -injective, by Proposition(2.2) Ă is almost Ɓ-injective semimodule.

(2)If Ă, Ɓ are indecomposable semimodules and Ă is generalized Ɓ-injective, then there are four cases:

Case1/ Ă =Ă⨁0 and Ɓ =0⨁ Ɓ, this means there are θ:0⟶ Ă and γ:0⟶ Ɓ, hence $\pi_{Ă}\alpha = \theta\,\pi'_0|_{Ų} = \theta\,\pi'_0 i_{Ų}$ and $\gamma\pi_0\alpha = \pi_{Ɓ}'|_{Ų} = \pi_{Ɓ}'\, i_{Ų}$ implies Ų ⊂ Ɓ$\oplus\gamma(0)$, $\alpha = 0$ and $\gamma = 0$ which is impossible.

Case2/ Ă =Ă⨁0 and Ɓ =Ɓ⨁0, this means there are θ: Ɓ ⟶ Ă and γ:0⟶0, then $\pi_{Ă}\alpha = \theta\,\pi'_{Ɓ}|_{Ų} = \theta\,\pi'_{Ɓ} i_{Ų}$ and $\gamma\pi_0\alpha = \pi'_0|_{Ų} = \pi'_0\, i_{Ų}$ implies that Ų ⊂ Ɓ$\oplus\gamma(0)$, $\alpha = \theta\, i_{Ų}$ and 0=0.

Case3/ Ă = 0⨁Ă and Ɓ=0⨁ Ɓ, then there are θ:0⟶0 and γ: Ă ⟶ Ɓ, hence Ų ⊂ Ɓ$\oplus\gamma$(Ă) with $\pi_0\alpha = \theta\,\pi'_0|_{Ų} = \theta\,\pi'_0 i_{Ų}$ and $\gamma\pi_{Ă}\alpha = \pi'_{Ɓ}|_{Ų} = \pi'_{Ɓ}\, i_{Ų} \Longrightarrow$ 0=0 and $\gamma\alpha = i_{Ų}$ implies α is monomorphism.

Case4/ Ă =0⨁ Ă and Ɓ =Ɓ⨁0, this means there are θ: Ɓ ⟶ 0 and γ: Ă ⟶ 0, hence Ų⊂ Ɓ$\oplus\gamma$(Ă) with $\pi_0\alpha = \theta\,\pi'_{Ɓ}|_{Ų} = \theta\,\pi'_{Ɓ} i_{Ų}$ and $\gamma\pi_{Ă}\alpha = \pi'_0|_{Ų} = \pi'_0\, i_{Ų}$ implies α = 0 which is impossible.

(3)From (1), it follows, whenever Ă is generalized Ɓ-projective semimodule and Ă, Ɓ are indecomposable, then either there is θ: Ɓ ⟶ Ă such that $\alpha = \theta\, i_{Ų}$, or there is γ: Ă ⟶ Ɓ such that $\gamma\alpha = i_{Ų}$ implies α is monomorphism and Ų is embedded in Ă.

(4)If there is no monomorphism α: Ų ⟶ Ă where Ų is any subsemimodule of Ɓ, then Case(3) is not satisfied, so Ă is a Ɓ -injective semimodule.

(5) If Ă is indecomposable and generalized self-injective semimodule, then by Corollary(3.2), Ă is almost self-injective semimodule.

Proposition 2.5: Let Ă be almost Ɓ-injective semimodule. Consider the following diagram:

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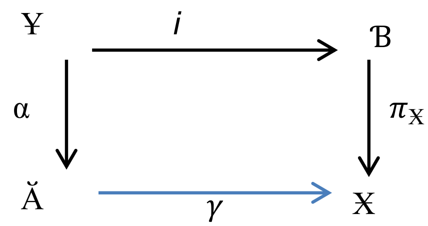


Put ǩ = ker(α). If the second case of definition(2.1, [1]) occurs, there is a proper direct summand Ḓ of Ă which contains ǩ. In particular, if ǩ $\leq_e$Ă, then the first case occurs.

Proof. Since the second diagram holds, we get a direct decomposition Ɓ=Ҳ⊕Ḓ and γ:Ă ⟶ Ҳ such that $\gamma\alpha = \pi_{\text{Ҳ}} i$, then $\pi_{\text{Ҳ}}$(ǩ) = $\pi_{\text{Ҳ}} i$(ǩ)= $\gamma\alpha$ (ǩ)= 0, so ǩ ⊆ ker($\pi_{\text{Ҳ}}$)= Ḓ, then Ḓ is a proper direct summand of Ă. If ǩ $\leq_e$Ɓ, we have ǩ ∩Ҳ=0 implies Ҳ=0 which is a contradiction. Hence the first case occurs. ■

Definition 2.6: A semimodule Ă is called essentially Ɓ-injective if, for every subsemimodule Ҳ of Ɓ, any homomorphism α:Ҳ⟶Ă where ker(α)is essential in Ҳ, can be extended to a homomorphism β: Ɓ⟶Ă.

Proposition 2.7: Let Ă and Ɓ be semimodules, whenever Ă is almost Ɓ-injective, then Ă is an essentially Ɓ-injective semimodule.

Proof. Assume that Ҳ is a sub-semimodule of Ɓ and α:Ҳ⟶Ă be any homomorphism with ker(α)is essential in Ҳ, then from Proposition(2.5) α can be extended to a homomorphism β:Ɓ⟶ Ă provided ker(α) $\leq_e$Ɓ (in fact, if Ұ is a complement of Ҳ in Ɓ, define ϑ:Ҳ⊕Ұ ⟶ Ă by $\vartheta(x + y) = \alpha(x)$ for every x in Ҳ and y in Ұ). Then ker(ϑ)=ker(α) + Ұ $\leq_e$ Ҳ⊕Ұ $\leq_e$Ɓ. So ker(ϑ) $\leq_e$Ɓ and hence ϑ can be extended to a homomorphism β:Ɓ⟶ Ă. Clearly, β is an extension of α. ■

From the previous, propositions, we conclude the following corollary.

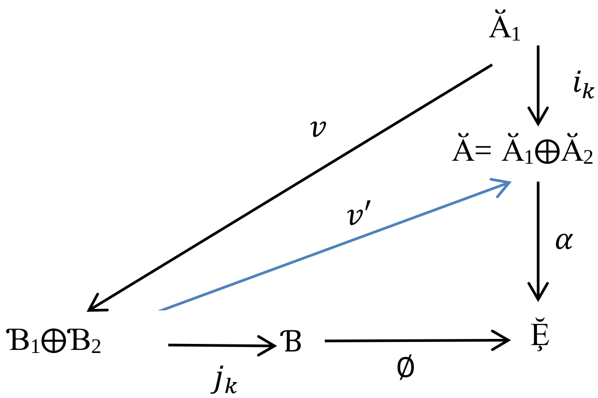
Corollary 2.8: If a semimodule Ă is generalized Ɓ-injective, Ă is essentially Ɓ-injective.

Proof. Since, if Ă is a generalized Ɓ-injective semimodule, then Ă is almost Ɓ-injective by proposition (2.2) and from proposition (2.7), we get Ă is an essentially Ɓ-injective semimodule.

3.Some Properties of Generalized Projective Semimodules

Dually, generalized projective semimodules can be defined, also the relationship of this concept with almost projective semimodules will be explained.

Definition 3.1: A semimodule Ă is called generalized Ɓ-projective when, for any homomorphism α: Ă⟶Ḝ, any epimorphism $\emptyset$: Ɓ⟶Ḝ, then decompositions occurs Ă =Ă$_1$⊕Ă$_2$ and Ɓ= Ɓ$_1$⊕Ɓ$_2$, a homomorphism v: Ă$_1$⟶ Ɓ$_1$ and an surjective homomorphism v': Ɓ$_2$⟶ Ă$_2$ satisfying $\emptyset j_1 v = \alpha i_1$ and $\alpha i_2 v' = \emptyset j_2$, where i_1, i_1 are the inclusion maps from Ă$_1$, Ă$_2$ into Ă respectively and j_1, j_2 are the inclusion maps of Ɓ$_1$, Ɓ$_2$ into Ɓ.



Remark 3.2:

(1)When Ã and Ɓ are indecomposable semimodules and a semimodule Ă is generalized Ɓ-projective, there are four cases:

Case1/ Ă=Ă⊕0 and Ɓ=0⊕Ɓ, this means there are v:Ă⟶0 and v':Ɓ⟶0, hence $\emptyset|_0 v = \alpha i_1$ implies $\alpha = 0$ which is impossible and $\alpha|_0 v' = \emptyset j_2$ implies 0 = 0.

Case2/ Ă=Ă⊕0 and Ɓ=Ɓ⊕0, this means there are v:Ă⟶Ɓ and v':0⟶0, then $\emptyset j_1 v = \alpha i_1 \Longrightarrow \emptyset v = \alpha$ and $\alpha|_0 v' = \emptyset|_0$ implies that 0=0.

Case3/ Ă= 0⊕Ă and Ɓ=0⊕ Ɓ, then there are v:0⟶0 and v':Ɓ⟶Ă, hence $\emptyset|_0 v = \alpha|_0$ implies 0=0 and $\alpha i_2 v' = \emptyset j_2 \Longrightarrow \alpha v' = \emptyset$ implies α is epimorphism.

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Case4/ Ă=0⊕ Ă and Ɓ=Ɓ⊕0, this means there are v: 0 ⟶Ɓ and v': 0 ⟶ Ă, hence $\emptyset j_1 v = \alpha|_0$ implies $\emptyset = 0$ which is impossible and $\alpha|_0 v' = \emptyset|_0$ implies 0 = 0.

(2)From (1), it follows that, whenever Ă is a generalized Ɓ-projective semimodule and Ă, Ɓ are indecomposable, then either there is v:Ă⟶Ɓ such that $\emptyset v = \alpha$, or there is v':Ɓ⟶Ă such that $\alpha v' = \emptyset$ implies α is epimorphism.

(3)If there is no epimorphism α: Ă⟶Ḝ where Ḝ is any semimodule which is homomorphic image of Ă, then Case(3) is not satisfying, so Ă is a Ɓ-projective semimodule.

A semimodule Ă is called a generalized self-projective semimodule if, Ă is generalized Ă-projective

Proposition 3.3: Suppose that Ă is a generalized Ɓ-projective semimodule, Ă is almost Ɓ-projective.

Proof: Suppose α: Ɓ ⟶ Ӿ is an epimorphism and δ:Ă⟶ Ӿ any homomorphism for any semimodule Ӿ. From hypothesis, decompositions occur Ă =Ă$_1$⊕Ă$_2$ and Ɓ= Ɓ$_1$⊕Ɓ$_2$, a homomorphism ϑ: Ă$_1$⟶ Ɓ$_1$ and an epimorphism ϑ': Ɓ$_2$⟶ Ă$_2$ satisfying $\alpha\vartheta = \delta|_{\breve{A}_1}$ and $\delta\vartheta' = \alpha|_{Ɓ_2}$. Thus Ă is almost Ɓ-projective. ■

Corollary3.4: Suppose that Ă is a generalized self-projective semimodule, then Ă is an almost self-projective semimodule.

Definition 3.5[7]: A semimodule Ă is noetherian (artinian), if every chain of subsemimodules of Ă satisfies ACC(ascending chain condition)(DCC(descending chain condition)). A semiring Ȑ is noetherian (artinian), if $Ȑ_{Ȑ}$ is noetherian (artinian) that is every chain of left ideals satisfies ACC(DCC).

Examples 3.6:

(1)A simple semimodule is noetherian and artinian.

(2)The semimodule ℕ over itself is noetherian but not artinian.

(3)The semimodule $\mathbb{Q}_{\mathbb{Z}}$ is not noetherian.

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Lemma 3.7: Let Ă be artinian (noetherian) semimodule, then Ă possible to express as a finite direct sum of indecomposable subsemimodules.

Proof. Assume Ă is artinian (notherian) semimodule. Assume that Ă unable to express into direct sum of indecomposable sub-semimodules, that is $Ă=Ă_1 \oplus Ă_2$ where $Ă_2$ cannot be written as a direct sum of indecomposable sub-semimodules. Write $Ă_2 = Ă_3 \oplus Ă_4$, where $Ă_4$ cannot de expressed into a direct sum of indecomposable sub-semimodules. Continuing this process, we have an infinite decreasing chain of subsemimodules of Ă, $Ă_2 \supseteq Ă_4 \supseteq \cdots \supseteq Ă_i \supseteq Ă_{i+1} \supseteq \cdots$, this contradiction with assumption, therefore Ă can be disassembled into a finite direct sum of indecomposable sub-semimodules. ■

Proposition 3.8: Let Ă be artinian, almost Ɓ- projective semi module, then Ă possibly expressed as as a finite direct sum of indecomposable almost Ɓ-projective sub-semimodules where Ɓ is any semimodule.

Proof. From the Lemma(3.7), Ă able to decompose into a finite direct sum of indecomposable sub-semimodules. Since Ă is almost Ɓ-projective, any summand of Ă is almost Ɓ-projective, so Ă is finite direct sum of indecomposable almost Ɓ-projective sub-semimodules.■

Dually, Proposition (3.8) can be applied for almost-injective semimodule as follows.

Proposition 3.9: Let Ă, Ɓ be semimodules and Ă is artinian, almost Ɓ-injective semimodule, then Ă possibly expressed as a finite direct sum of indecomposable almost Ɓ-injective sub-semimodules.

QI-ring is defined by many authors as [8], can be generalized to define QI-semiring as follows:

Definition 3.10: A semiring Ȑ is said to be QI-semiring, if every quasi-injective Ȑ-semimodule is injective. It is clear that any semi simple semiring is QI.

Proposition 3.11: If Ȑ is QI-semiring, then the direct sum of two quasi-injective Ȑ-semimodule is quasi-injective.

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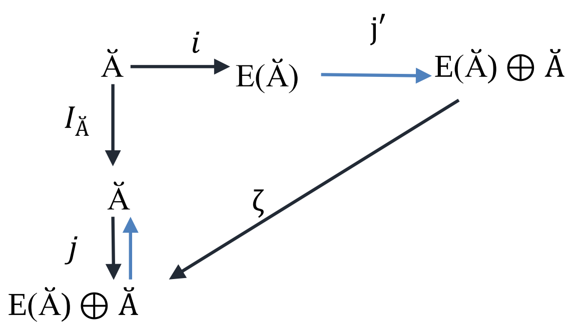
Proof. Let A and B be two quasi-injective semimodules, then they are injective from assumption, then A⊕B is injective semimodule since every injective semimodule is quasi- injective, and then the direct sum A ⊕B is quasi-injective . ■

Proposition 3.12: If the direct sum of two quasi-injective Ȓ-semimodules $Ă_1, Ă_2$ with injective hulls $E(Ă_1)$, $E(Ă_2)$ is always quasi-injective, then Ȓ is QI-semiring.

Proof. Let Ă be a quasi-injective semimodule. Consider the following diagram:



Where i is the inclusion map and j, j' are the injection maps from Ă, E(Ă)

respectively to E(Ă)⊕Ă. Since E(Ă)⊕Ă is quasi-injective semimodule, there is endomorphism ζon E(Ă)⊕Ă such that $\zeta j' i = j I_Ă$. Take $\theta = \pi_Ă \zeta\, j'$ hence $\theta i = I_Ă$, then Ă is isomorphic to summand of E(Ă), and hence it is an injective semimodule, this implies that Ȓ is QI-semiring. ■

The following result in[9] is proved for the case of modules and it can be extended to semimodules with similar proof.

Lemma 3.13: Let Ă be semimodule with injective hull E(Ă), then Ă is quasi-injective if and only if it is fully invariant in the injective hull of Ă.

Proposition 3.14: If Ȓ is QI-semiring, then every injective Ȓ-semimodule Ă with injective hull satisfies the property "every fully invariant subsemimodule of Ă is a direct summand".

Proof. Let Ă be injective semimodule and Ɓ be fully invariant subsemimodule of Ă, it suffices to show that Ɓ is a quasi-injective, let θ:E(Ɓ)→E(Ɓ), then θcan be extended to endomorphism

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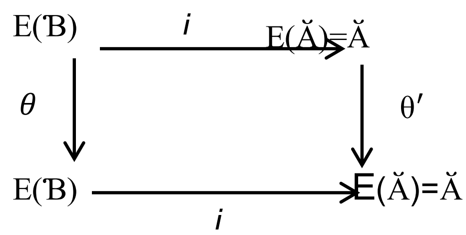
θ':E(Ă)→E(Ă), since Ɓ is fully invariant in Ă, then θ(Ɓ)= θ'(Ɓ)≤Ɓ, therefore by Lemma(3.9), Ɲ is quasi-injective semimodule. As the following diagram:



اكتب المعادلة هنا.

The converse of the various proposition is true, that is:

Proposition 3.15: If every injective Ȑ-semimodule Ă with injective hull satisfies the property "every fully invariant subsemimodule of Ă is a direct summand", then Ȑ is QI-semiring.

Proof. Assume that Ă is a quasi-injective semimodule, from Lemma(3.13) Ă is fully invariant in E(Ă), so Ă is a summand of E(Ă) by hypothesis. Therefore Ă=E(Ă). Therefore Ȑ is a QI-semiring. ■

Definition 3.16: A semiring is called AQI-semiring, if every almost self-injective semimodule is injective.

Examples 3.17:

(1)Every QI-semiring is AQI-semiring since every quasi-injective semimodule is almost self-injective semimodule.

(2)Every semi simple semiring is AQI-semiring.

Proposition3.18: If Ȑ is AQI-semiring, then the direct sum of two almost self-injective semimodule is almost self-injective.

Proof. Let A and B be two almost self-injective semimodules, then they are injective from assumption, then A⊕B is injective semimodule since every injective semimodule is almost-self-injective, then the direct sum A ⊕B is almost-self-injective . ■

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Proposition 3.19: If the direct sum of two almost self-injective Ȓ-semimodules $Ă_1, Ă_2$ with injective hulls $E(Ă_1)$, $E(Ă_2)$ is always almost self-injective, then Ȓ is AQI-semiring.

Proof. Similar to Proposition(3.18). ■

Conflict of interests.

There are non-conflicts of interest.

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